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Optimal Investment Problems under Market Frictions

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To Daniela
and to our baby

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Introduction

The main subject of this dissertation are general optimization problems in markets with proportional transaction costs, with possibly incomplete information and constraints on strategies. The framework considered encompasses different problems, such as derivatives hedging, utility maximization from terminal wealth, and problems of consumption and investment.

In the last decade, these problems have raised considerable interest, and have been attacked with different approaches. Stochastic control theory has been employed by Davis and Norman [DN90] for the problem of intertemporal utility maximization under transaction costs and by Davis, Panas, and Zariphopoulou [DPZ93] for options hedging in the Black-Scholes model with transaction costs. Stochastic control techniques have the advantage of characterizing the value functions of optimization problems as weak solutions of Hamilton-Jacobi-Bellman PDEs, but they are applicable only with Markov assets processes.

Such limitation is overcome by the convex duality approach, introduced in the case of Itô asset processes by Karatzas, Lehoczky and Shreve [KLS87] in the context of frictionless complete markets. The case with incomplete information has been covered by Karatzas, Lehoczky, Shreve and Xu [KLSX91], while in the semimartingale case Kramkov and Schachermayer [KS99] have determined necessary and sufficient conditions on utility functions for the existence of optimal solutions to the problem of utility maximization from terminal wealth.

Within the convex duality framework, Cvitanic and Karatzas [CK96] first addressed hedging and optimization problems under transaction costs, in the case of Itô asset processes. In the semimartingale case, Deelstra, Pham and Touzi [DPT00] have shown the existence of solutions to problems of utility maximization from terminal wealth, relaxing the regularity assumptions on utility functions. In a continuous semimartingale setting, Kamizono [Kam01] has recently shown the existence of solutions to a class of intertemporal utility maximization problems under transaction costs.

Convex duality arguments become especially complex in presence of transaction costs. In fact, the first step in the dual formulation consists in an *optional decomposition* result (see for instance Kramkov [Kra96]), which translates the admissibility of strategies into expectation conditions under a set of martingale measures. This result is also sensitive to constraints on strategies, which usually call for a separate nontrivial treatment (as for instance in Föllmer and Kramkov [FK97]). In addition, the coverage of nondifferentiable utility functions, which becomes necessary in presence of transaction costs, requires the use of tools of nonsmooth convex analysis, even for the existence of solutions.

Here we take a different approach, which avoids the dual formulation altogether and deals with the original problem directly. This means that we do not need an optional decomposition result in continuous time under transaction costs (which is, to our knowledge, still an open area), and that no particular care will be necessary for utility functions which lack regularity.

One of the main points that we make is that existence problems are in fact *easier* when transaction costs are introduced. This is a consequence of the regularity of admissible strategies, which are forced to a class of finite variation processes. Under a general no-arbitrage condition, it turns out that this class enjoys convenient compactness properties, which imply the existence of solutions for a large class of optimization problems. Once this result is established, introducing constraints on strategies adds little complexity, as long as these constraints are convex and closed.

The dissertation is organized as follows: In Chapter 1 we describe in details the class of market models considered, giving particular emphasis to the different sets of consistent assumptions. Essentially, a well-posed optimization problem requires three basic choices: *i*) an arbitrage-free asset process, *ii*) a set of admissible strategies and *iii*) an optimization objective. Of course, the first two choices are strictly connected, since arbitrage opportunities are a subset of admissible strategies.

Even in frictionless markets, these choices are not unique. In fact, in classical Finance Theory it is common to consider the space of strategies with a finite variance payoff, which are natural in presence of quadratic risk criteria. On the other hand, the mainstream choice in Mathematical Finance is to define admissible strategies by means of *budget constraints*, which do not depend on agents' views (represented by the physical probability measure P). Since neither of these choices is superior in generality, we consider the optimization problems associated to both of them.

The optimization objective is generally given by the agent's preferences, which can be modeled via *utility functions*, *coherent risk measures*, or with other risk criteria (i.e. *expected shortfall*). We consider a wide class of *convex decreasing risk functionals* which encompasses most of these cases.

Since in presence of transaction costs the equivalence between absence of arbitrage and the existence of (local) martingale measures is no longer true, in Chapter 2 we discuss this issue and establish some simple sufficient conditions, which allow to consider models that may allow arbitrage without transaction costs. These conditions turn out to be very similar to recent equivalent no-arbitrage conditions under transaction costs in discrete time.

In Chapter 3 we consider optimization problems defined over sets of strategies leading to p-integrable payoffs, a generalization of the classical setup of mean-variance optimization. Assuming that assets are continuous semimartingales, we show the existence of optimal strategies for *convex decreasing risk functionals*. This result is obtained through a direct method technique, identifying a convergence which makes minimizing sequences compact, and the risk functional lower semicontinuous. The existence results are then easily adapted to include convex constraints, such as limits on short-selling and budget constraints. However, although the class of *convex decreasing risk functionals* features natural economic properties, it does not include the mean-variance criterion, which is the most relevant application of strategies with p-integrable payoffs.

In fact, we treat this case separately in Chapter 4. As we shall show, it turns out that mean-variance optimization in presence of transaction costs leads to paradoxical consequences, arising from the combination of penalizing gains with the variance criterion and the ability to dissipate wealth through transaction costs. In fact, we show that the mean-variance hedging problem with transaction costs is equivalent to the minimization of the expected squared shortfall, which is a convex decreasing risk functional.

In Chapter 5 we take up the modern approach of excluding arbitrage through budget constraints, thus relaxing all integrability conditions. As the semimartingale property is not natural in presence of transaction costs, we also consider more general asset processes, imposing only one of the no-arbitrage criteria discussed in Chapter 2. For technical reasons, in this context we consider quasi left-continuous asset processes. This condition basically amounts to assuming that jumps are totally inaccessible, and is satisfied by most financial models considered in the literature (e.g. for all jump-diffusions).

As in Chapter 3, the existence of strategies is obtained through compactness and semicontinuity, but in this case we need to consider a much weaker convergence, which is invariant up to changes to equivalent measures. Since the proofs can be simplified for continuous and for semimartingale asset processes, we show this two cases separately. We then extend the existence results to convex constraints, and apply them to the general utility maximization problem.

Since the results in this chapter rely on a *compact no-arbitrage* criterion, which is not a necessarily satisfied by *any* arbitrage-free model with transaction costs, we show with a counterexample that the compactness result may not hold under a plain *no-arbitrage* condition.

Finally, we consider intertemporal optimization problems, where economic agents can make both consumption and investment decisions. Since cumulative consumption can be described by an increasing process, and strategies with transaction costs are necessarily of finite variation, it turns out that the same compactness results can be applied directly to this setting. Hence we obtain the existence of solutions for a class of intertemporal problems, where the objective functional depends on the cumulative consumption process, while the trading strategy enters only in the budget constraint.

Chapter 1

The Model

We start describing in details our model of a financial market with frictions. As usual in Mathematical Finance, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions, and $\mathcal{F} = \mathcal{F}_T$.

In this market we have a riskless asset and d risky assets. The riskless asset is used as numeraire, hence it is assumed identically equal to 1. The prices of the risky assets are given by an $\mathbb{R}^d$-valued process $X = ((X^i_t)^d_{i=1})_{0\leq t\leq T}$, adapted to the filtration $\mathcal{F}_t$.

All assets are arbitrarily divisible, and trading on the risky assets is subject to proportional transaction costs, which may vary across assets. We denote by k_i the cost associated to the purchase or sale of a unit of the i-th asset, so that the transaction cost for one share of the i-th asset at time t is $k_i X^i_t$. We denote by k the vector $(k_1, \ldots, k_d)$.

In general, we can expect k to depend both on t and ω, reflecting changing liquidity conditions at different times and circumstances. Hence, we will allow k to be an adapted, strictly positive stochastic process, and will discuss further assumptions when needed.

In this market, an agent is endowed with an initial capital c (units of numeraire), and trades in all assets in order to maximize some objective, without counting on external income sources, such as wages. We have in mind two main classes of problems:

i) objectives depending on terminal wealth only, as *utility maximization* *from terminal wealth* and *hedging of contingent claims*. In this class, the agent only needs to make investment decisions, and we obtain an optimization problem on a class of *admissible strategies*.

ii) intertemporal objectives, which essentially include all problems of maximization of utility from consumption along the time interval $[0, T]$. In this class, the agent needs to make both consumption and investment decisions.

1.1 The gain and portfolio processes

Since we are going to consider asset processes which are not necessarily semimartingales, first we need to extend the definition of gain process to this setting. The Bichteler-Dellacherie Theorem (see for instance [Pro90]) characterizes semimartingales as the largest class of integrators for general predictable strategies, therefore we will have to restrict the class of integrands.

In any reasonable market model, it is generally accepted that trading gains should be finite almost surely. Introducing proportional transaction costs, it is then natural to assume that trading volume remains finite almost surely, in order to avoid the possibility of infinitely negative wealth.

In mathematical terms, this amounts to consider only finite variation strategies, which can be integrated *path by path*:

Definition 1.1 (Dellacherie and Meyer [DM82], 8.1).
Let $\theta : \mathbb{R}^+ \mapsto \mathbb{R}$ be a function of locally bounded (e.g. finite) variation, and X a cadlag function. Then we define the integral of θ_- with respect to X as:

$$(\theta \cdot X)_t = \int_{[0,t]} \theta_{s^-} dX_s = \theta_{t^+} X_t - \theta_{0^+} X_0 - \int_{(0,t]} X_s dD\theta_{s^+} \tag{1.1}$$

Given a function of bounded variation θ, we denote by $D\theta$ its derivative in the sense of distributions, which is a Radon measure, by $|D\theta|$ the total variation measure associated to $D\theta$, and by $|D\theta|_t = |D\theta|[0, t]$. These definitions trivially extend componentwise to vector-valued strategies $\theta_t = (\theta_{1,t}, \dots, \theta_{d,t})$ as $D\theta_t = (D\theta_{1,t}, \dots, D\theta_{d,t})$, $|D\theta|_t = (|D\theta_1|_t, \dots, |D\theta_d|_t)$. Denote also $\|D\theta\|_t = \sum_{i=1}^d |D\theta_i|_t$. Conventionally, we assume that $\theta_{0^-} = 0$, so that (1.1) can be written as:

$$(\theta \cdot X)_t = \theta_{t^+} X_t - \int_{[0,t]} X_s dD\theta_{s^+}$$

As a result, by the *self-financing* condition, the liquidation value of the portfolio at time t is:

$$V_t^c(\theta) = c + (\theta \cdot X)_t - \sum_{i=1}^{d} k_i \left(\int_{[0,t]} X_s^i d|D\theta_i|_s + |\theta_{i,t}| X_t^i \right) \tag{1.2}$$

where the terms in the right-hand side represent respectively the initial capital, the trading gain, the cost of the trading strategy, and the cost of the final liquidation of the position. With an abuse of notation, we identify the vector k with the $d \times d$ diagonal matrix with elements $(k_1, \ldots, k_d)$ so that we can rewrite the above expression as

$$V_t^c(\theta) = c + (\theta \cdot X)_t - \int_{[0,t]} k_s X_s \cdot d|D\theta|_s - k_t X_t \cdot |\theta_t| \tag{1.3}$$

When consumption is present, this formula becomes:

$$V_t^c(\theta, C) = c + (\theta \cdot X)_t - \int_{[0,t]} k_s X_s \cdot d|D\theta|_s - k_t X_t \cdot |\theta_t| - C_t \tag{1.4}$$

where C_t is an adapted, right-continuous process, representing cumulative consumption in the interval $[0, t]$.

Remark 1.2. The pointwise definition of variation can be modified into the following (much less intuitive), which is invariant up to sets of Lebesgue measure zero:

$$|D\theta|(\omega) = \sup_{\substack{\phi \in C_c^1(0,T) \\ \|\phi\|_\infty \leq 1}} \int_{[0,T]} \theta_s(\omega) \phi'(s) ds$$

In fact, it can be shown that for each θ there exists a representative such that the expression above coincides with the (generally higher) pointwise variation.

1.2 Arbitrage and admissible strategies

At this point, in a frictionless market we should add restrictions on the agent's trading strategies, in order to exclude the possibility of arbitrage opportunities, which make most optimization problems trivial. A natural question is then if this issue disappears with the introduction of transaction costs. In other words, we have to check if there is an analogous of *doubling strategies* (see Harrison and Pliska [HP81]) in presence of proportional transaction costs.

We define arbitrage as follows:

Definition 1.3. *A strategy θ is an* arbitrage opportunity *if, for some t, $V_t^0(\theta) \geq 0$, $P(V_t^0(\theta) > 0) > 0$.*

The next example provides a positive answer to the above question:

Example 1.4. Consider a discrete-time binomial model with infinite horizon, which can obviously be embedded in a piecewise constant continuous-time model in the interval $[0, T]$ by the time-change $n \mapsto T - \frac{1}{n}$.

We have only one risky asset X, such that $X_0 = 1$. At each step n, X_{n+1} can jump either to $X_n(1 + \varepsilon)$ with probability p, or to $X_n(1 - \varepsilon)$ with probability $1 - p$. Transaction costs are equal to k times the amount transacted.

We denote an agent's position as (C, S), where C is the amount of cash (units of riskless asset) and S is the amount of stock holdings (as opposed to the number of shares held). Hence, the liquidation value of (C, S) is given by $C+S-k|S|$. From any position (C, S), we want to setup a portfolio such that, if the market goes up (i.e. $X_{n+1} = (1 + \varepsilon)X_n$) then the liquidation value of our position is a given positive constant G. In other words, we want to move D units from C to S, so that our position becomes $(C - D - k|D|, S + D)$, choosing D such that:

$$C - D - k|D| + (S + D)(1 + \varepsilon) - k|S + D|(1 + \varepsilon) = G$$

When both D and $S + D$ are positive, we obtain:

$$C - (1 + k)D + (S + D)(1 + \varepsilon)(1 - k) = G$$

and hence:

$$D = \frac{G - C - S(1+\varepsilon)(1-k)}{\varepsilon - k(2+\varepsilon)} \tag{1.5}$$

which is positive, provided that k is small enough, and that $G > C + S(1 + \varepsilon)(1-k)$. In particular, this the case when $C + S(1+\varepsilon)(1-k) \leq 0$ and $G > 0$.

Now we can construct the following arbitrage: starting from the position $(0,0)$, we choose D as in (1.5), obtaining a position (C_0, S_0) with negative liquidation value (because of the transaction cost). If the market goes up, we liquidate a profit of G. If not, we get to the position $(C_1, S_1) = (C_0, S_0(1-\varepsilon))$, and we have that:

$$C_1 + S_1(1+\varepsilon)(1-k) = C_0 + S_0(1-\varepsilon^2)(1-k) < C_0 + S_0(1-k) < 0$$

Therefore, we can still apply (1.5) to increase our position in the stock, so that an upward market move at the next step will still guarantee a profit of G. Iterating this procedure, we clearly obtain an arbitrage strategy.

The previous example shows that, even in presence of transaction costs, we need to prevent arbitrage by some constraint. Indeed, this can be achieved by different means, and we shall consider two of them:

- *integrability conditions*
- *budget constraints*

1.2.1 Integrability conditions

Doubling strategies are known since the dawn of Probability Theory (if not earlier), well before the rise of Mathematical Finance. In fact such strategies were called *martingales* among gamblers of horse races, and indeed the wealth process in such a strategy is a martingale, provided that the game is fair. Moreover, this is the classical example of a non *uniformly integrable* martingale, and this is precisely the feature that leads to arbitrage.

This observation hints that arbitrage can be avoided enforcing uniform integrability. The simplest way to achieve this is to ensure that the wealth process is bounded in L^p with $p > 1$, which of course implies uniform integrability by De la Vallée Pussin criterion.

This approach goes back to Harrison and Kreps [HK79] and Kreps [Kre81] and it is most natural in the context of *mean-variance hedging* (see for instance Schweizer [Sch96] and Rheinländer and Schweizer [RS97]), where integrability is granted by the optimization objective.

A pleasant feature of this setting of *attainable claims with p-th moments* (as defined by Delbaen and Schachermayer [DS96]) is the existence of a natural duality between the set of p-integrable contingent claims and the set of martingale measures with p'-integrable density, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1.5. *For each $q \geq 1$, define the sets of martingale measures:*

$$\mathcal{M}_q^e(P) = \{Q \sim P : \frac{dQ}{dP} \in L^q(P), X \text{ is a } Q\text{-local martingale}\}$$

In order to exploit this feature, in this context we shall make the following assumptions on the asset process X:

Assumption 1.6. *X is a continuous semimartingale which admits at least a local martingale measure.*

In the case of frictionless markets, the natural space of strategies is given by:

$$\Theta^p = \{\theta : \theta\ \mathcal{F}_t\text{-predictable, } (\theta \cdot X)_T \in L^p(P)\}$$

where p is generally related to the optimization objective (for instance, $p = 2$ in the case of *mean-variance hedging*). However, the problem with the space Θ^p is that it is generally not closed under the topology induced by the map $G_T : \theta \mapsto (\theta \cdot X)_T$, unless additional assumptions are made on X. In fact, for $p = 2$, Delbaen, Monat, Schachermayer, Schweizer and Stricker [DMS$^+$94] have established a necessary and sufficient condition.

Unfortunately, this condition fails to hold for some stochastic volatility models (see for instance Biagini, Guasoni and Pratelli [BGP00] for an example), suggesting that the choice of the space Θ^p may not be satisfactory.

In this spirit, Delbaen and Schachermayer [DS96] have proposed a different space K_p with better closure properties:

Definition 1.7. *Let K_p^s be the set of bounded simple integrals with respect to X. K_p denotes the closure of K_p^s in the norm topology of $L^p(P)$.*

In the case of continuous processes, the properties of the space K_p are summarized by the following theorem, which is a short version of the main result in [DS96]:

Theorem 1.8. *Let $1 \le p \le \infty$, and $p' = \frac{p}{p-1}$. If X is a continuous semimartingale locally in $L^p(P)$ such that $\mathcal{M}^e_{p'}(P) \neq \emptyset$, and $f \in L^p(P)$, the following conditions are equivalent:*

i) $f \in K_p$;

ii) There exists a X-integrable predictable process θ such that $G_t(\theta)$ is a uniformly integrable Q-martingale for each $Q \in \mathcal{M}^e_{p'}(P)$, and $G_t(\theta)$ converges to f in the $L^1(Q)$ norm (as t converges to T);

iii) $E_Q[f] = 0$ *for each* $Q \in \mathcal{M}^e_{p'}(P)$.

In chapter 3 we take up this approach, and define the space Θ^p_C of p-integrable strategies with transaction costs. We then study optimization problems in this space, and show the existence of solutions.

1.2.2 Budget constraints

The main drawback of integrability conditions is that L^p spaces depend critically on the choice of the agent measure P, while in most cases trading strategies depend on the institutional characteristics of the market, rather than on agents' views. Besides, it is seldom possible to identify clearly the measure P and the tail behavior of the asset process X, let alone the space of p-integrable strategies.

These and other objections have led many authors to consider instead *budget constraints*, which only depend on the equivalence class of P (that is, the class of *impossible* events), a notion far more easily agreed upon.

As proposed by Harrison and Pliska [HP81], a *budget constraint* requires that wealth remains bounded from below at all times.

Hence in our context of transaction costs we have the following definition of *admissibility*:

Definition 1.9. *A predictable finite variation process θ is called* admissible *if, for some $c > 0$ and for all t we have that $V^c_t(\theta) \ge 0$ a.s. .*

We denote by $\mathcal{A}_c^k = \{\theta : V_t^c(\theta) \geq 0\}$ *the class of admissible strategies with initial capital* c *and proportional transaction costs* k*.*

In the frictionless case, it is well-known that a market is arbitrage-free (it satisfies the (NFLVR) condition, to be precise) if and only if there exists a probability Q, equivalent to P, such that X is a local martingale under Q.

In presence of transaction costs, the existence of a martingale measure remains of course a sufficient condition. However, one can imagine consistent arbitrage-free processes which have no martingale measures, and even ones that are not semimartingales.

In order to allow for such processes, we make the following assumption on the asset process X:

Assumption 1.10. X *strictly positive, adapted cadlag process.*

Of course, such a broad class of processes cannot possibly exclude the presence of arbitrage opportunities, and some additional condition is required. We discuss these conditions in chapter 2, where some no-arbitrage criteria are established.

Some results will also require the quasi left-continuity property, which amounts to assuming that the timing of jumps is unpredictable:

Definition 1.11. *A process* X *is* quasi-left continuous *if* $X_\tau = X_{\tau^-}$ *for all predictable stopping times* τ*.*

The next proposition shows that for quasi-left continuous processes we only need to consider left-continuous strategies.

Proposition 1.12. *Let* X *be a quasi-left continuous process, and* θ *a predictable, finite variation process. Then we have that:*

$$\int_{[0,t]} \theta_s dX_s = \int_{[0,t]} \theta_{s^-} dX_s \quad a.s.$$

Proof. Since X is a cadlag adapted process, $\{\Delta X \neq 0\} = \bigcup_k [\![\tau_k]\!]$, where τ_k is a sequence of stopping times with disjoint graphs, and each of them is either predictable or totally inaccessible (see for instance Dellacherie and

Meyer [DM78]). When X is quasi-left continuous, up to a null set we can assume that all τ_k are totally inaccessible. It follows that:

$$\int_{[0,t]} \theta_s dX_s - \int_{[0,t]} \theta_{s^-} dX_s = \sum_{\tau_k} \Delta\theta_{\tau_k} \Delta X_{\tau_k}$$

Now, since θ is a predictable process of finite variation, it admits left and right limits for all t. Hence the jump process $\Delta\theta_t = \theta_t - \theta_{t^-}$ is itself predictable, and we can define:

$$V_t = \sum_{s \leq t} \Delta\theta_s$$

which is a predictable, cadlag process. Hence it follows (see Dellacherie and Meyer [DM78], Chapter IV, Theorem 88B) that the jump set $\{\Delta\theta \neq 0\} = \{\Delta V \neq 0\} = \bigcup_k [\![\sigma_k]\!]$ where σ_k is a sequence of predictable stopping times.

However, this means that $\Delta\theta_{\tau_k}$ is indistinguishable from the null process for all k, and the proof is complete. □

1.3 Preferences

After defining the characteristics of tradable assets and the set of strategies available to the agent, we now turn to the optimization objective, completing the description of the problem setup.

As mentioned at the beginning of the chapter, we have in mind problems of utility maximization and risk minimization at a fixed horizon and in an intertemporal setting, and we will treat these two cases separately.

1.3.1 Fixed horizon

As mentioned at the beginning of this chapter, many problems in Mathematical Finance require the optimization of an objective which involves the agent's position only at a fixed time. For example, in the classical "retirement problem" (see for instance [Mer69]) an economic agent maximizes the expected utility of his terminal wealth at a fixed time T (the retirement date):

$$\max_{\theta \in \mathcal{A}_c^k} E\left[U(V_T^c(\theta))\right]$$

Another typical example is the problem of hedging contingent claims with a fixed expiration (possibly path-dependent). In this case, the option writer has a random liability H at time T, and wants to set up a portfolio which minimizes the risk of his position. For instance, the agent may choose to minimize the expected shortfall:

$$\min_{\theta \in \mathcal{A}_c^k} E\left[(V_T^c(\theta) - H)^-\right]$$

Both these problems share the dependence on θ only through the liquidation value of the portfolio $V_T^c(\theta)$ at a fixed date, and possibly on a further random variable H. Also, both of them can be seen as minimization problems of the form:

$$\min_{\theta \in \mathcal{A}_c^k} \rho(V_T^c(\theta) - H)$$

where ρ is a functional assigning to each random variable X a real number $\rho(X)$ representing the risk perceived by the agent with such a terminal wealth. In order to embed several optimization problems of this kind into a single framework, we shall deal with risk functionals ρ satisfying some general assumptions.

For technical reasons, the assumptions on ρ will be slightly different in the two contexts of *integrability conditions* and *budget constraints*.

In the case of *integrability conditions* we shall consider mappings ρ satisfying the following:

Definition 1.13. *We denote by a* convex decreasing risk functional in L^p *a function* $\rho : L^p(\Omega, P) \mapsto \mathbb{R} \cup \{+\infty\}$*, satisfying the following properties:*

i) ρ *is convex;*

ii) if $X \leq Y$ *almost surely, then* $\rho(X) \geq \rho(Y)$ *;*

iii) ρ *has the Fatou property. Namely, if* $X_n \to X$ *a.s., then*

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$$

In the case of *budget constraints* we consider the following:

Definition 1.14. *We denote by a* convex decreasing functional *a function* $\rho : L^0 \mapsto \mathbb{R} \cup \{+\infty\}$, *with the following properties:*

i) ρ *is convex;*

ii) *if* $X \leq Y$ *almost surely, then* $\rho(X) \geq \rho(Y)$;

iii) *If* $X_n \to X$ *a.s., and* $X_n \geq -a$ *a.s. for some* $a > 0$, *then*

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$$

The choice of the above definitions is motivated by the following economic justifications:

i) This property reflects the usual condition of *risk aversion* and can also be seen as a weak principle of *diversification*. Essentially, it requires that an agent has no incentive in taking risks which offer no return.

ii) This assumption expresses a natural property of *weak nonsatiation*, since it requires that if a random wealth is almost surely higher than another, is should not be perceived as riskier.

iii) This is a technical assumption, devised in order to exclude risk functionals ρ with purely finite additive properties. A thorough discussion on this issue can be found in Delbaen [Del00], in the context of *coherent risk measures.*

Remark 1.15. In definition 1.13, the critical point is that we require the Fatou property for *any* sequence of random variables representing the terminal wealth of a trading strategy. On the contrary, in the definition of σ-additive coherent risk measure (see Delbaen [Del00]) the same property is required only for sequences bounded in L^∞.

For $\rho(X) = E[X^-]$ (i.e. minimizing the *shortfall*), *iii*) follows from a straightforward application of Fatou's Lemma. On the other hand, this property does not hold for general coherent risk measures, unless additional restrictions on trading strategies are made.

This difficulty disappears in presence of *budget constraints*, which force all strategies to be bounded. In fact, in definition 1.14 the Fatou property is only required for sequences uniformly bounded from below. This means that it includes all σ-additive *coherent risk measures* as defined by Delbaen [Del00].

1.3.2 Intertemporal setting

This class of problems encompasses various types of intertemporal utility maximization from consumption. As opposed to the fixed horizon case, in this case the agent's objective is a result of both consumption and investment decisions, with an obvious tradeoff between the two.

The representation of intertemporal preferences has recently been studied by Hindy, Huang and Kreps [HHK92] in a deterministic model, and by Hindy and Huang [HH92] in a stochastic setting. They have proposed a class of functionals satisfying the natural economic requirements that similar consumption patterns at nearby dates should be close substitutes, and that utility should depend on past as well as present consumption.

Following Hindy, Huang and Kreps [HHK92], we define the space of consumption plans:

$$\mathcal{C} = \{C : C_t \text{ adapted, right-continuous and increasing, and } C_{0^-} = 0\}$$

where C_t represents the cumulative consumption up to time t. On this space, we shall consider optimization problems of the type:

$$\begin{cases} \max\limits_{\theta, C} \rho(C) \\ V_t^c(\theta, C) \geq 0 \text{ a.s. in } dtdP \end{cases}$$

where $V_T^c(\theta, C)$ is defined by (1.4).

On the objective function ρ we shall make the following assumptions, analogous to those made in the fixed horizon case:

Definition 1.16. *We denote by an* intertemporal convex decreasing functional *a function* $\rho : \mathcal{C} \mapsto \mathbb{R} \cup \{+\infty\}$, *with the following properties:*

i) *ρ is convex;*

ii) *if $C \leq D$ a.s. in $dtdP$, then $\rho(C) \geq \rho(D)$;*

iii) *If $C_n \to C$ a.s. in $dtdP$, then $\rho(C) \leq \liminf_{n\to\infty} \rho(C_n)$.*

Remark 1.17. The right-continuity assumption in the definition of consumption plans is a mere convention, and could well be replaced by left-continuity. In fact, the only relevant properties of consumption plans depend on the positive measure dC_t, which would probably be a more natural definition. Nevertheless, we keep to the standard definition in terms of cumulative consumption since it fits well in our framework, where also the control variable θ is a finite-variation process.

1.4 Notes

Remark 1.18. The most general setting for problems with transaction costs is probably the one used in Kabanov et al. [Kab99, KL02, KRS01, KS01]. In those papers, a model of $d + 1$ currencies (assets) $S = (S^0, \dots, S^d)$ is considered, where exchanges between assets are settled as follows: in order to transfer a unit of account (which generally does not correspond to any currency) to asset j, we have to withdraw $1 + \lambda^{ij}$ units from some other asset i. Transaction costs are hence represented by a matrix $\Lambda = \{\lambda^{ij}\}_{ij}$ with nonnegative entries and such that $\lambda^{ii} = 0$ for all i.

This model has the following features:

i) there is no distinction between risky and riskless assets;

ii) it allows for exchanges between all pairs of assets;

iii) transaction costs for buying and selling can be different.

The model described in this chapter can be embedded into this framework as follows: S_0 is assumed identically 1, $\lambda^{0i} = k_i$, $\lambda^{i0} = \frac{k_i}{1-k_i}$ for all $i > 0$ and $1+\lambda^{ij} > (1+\lambda^{i0})(1+\lambda^{0j})$ for all $i, j > 0$. With these assumptions, exchanges between risky assets are always suboptimal, and transaction costs correspond to those in our model.

However, it is worthwhile to make the following observations:

i) In our model a trading strategy is represented by a vector-valued process θ_t. On the contrary, allowing for exchanges among all assets requires the use of a matrix-valued process L_t^{ij}, representing the cumulative amounts of transfers from asset i to asset j.

ii) The use of a transfer process L_t^{ij} allows for suboptimal strategies, where L_t^{ij} and L_t^{ji} may simultaneously increase. In other words, it is allowed to buy and sell one asset at the same time. The elimination of these strategies hence requires the introduction of a constraint.

iii) When purchases and sales generate different transaction costs, we have a buying (ask) price $(1+\lambda)X_t$ and a selling (bid) price $(1-\mu)X_t$. If we replace the asset X_t with $\tilde{X}_t = (1 + \frac{\lambda-\mu}{2})X_t$ and consider transaction costs equal to $k = \frac{\lambda+\mu}{2}$, then the bid and ask prices remain the same. This shows that we can easily reduce to this simpler case, up to a scaling of the asset prices.

iv) In most real markets (including foreign exchange) all transactions take place between a reference asset (currency for stock or commodity markets, and a reference currency, such as dollars, for foreign exchange transactions). Also, transaction costs may vary across assets with different liquidity, but are generally equal for buying and selling transactions. Hence the relevance of the model considered here for applications.

Finally, note that our definition of arbitrage strategy (1.3) corresponds to the definition of strong arbitrage in Kabanov et al. [KS01, KRS01]. As a result, by an *arbitrage-free* market we shall mean one which satisfies the $\mathbf{NA}^s$ condition in the above papers.

Remark 1.19. Recently, Schachermayer [Sch01a] has proposed a different framework for models with transaction costs, introducing a bid-ask matrix $\Sigma = \{\sigma^{ij}\}_{0\leq i,j\leq d}$ which is related to the previous notation as follows:

$$\sigma^{ij} = (1+\lambda^{ij})\frac{X^j}{X_i}$$

Hence, our model is obtained choosing, for all $i, j > 0$:

$$\begin{aligned}
\sigma^{0i} &= (1+k^i)X^i \\
\sigma^{i0} &= \frac{1}{(1-k^i)X^i} \\
\sigma^{ij} &> \sigma^{i0}\sigma^{0j}
\end{aligned}$$

Since the formulation with the bid-ask matrix Σ is equivalent in terms of generality to that with the transaction costs matrix Λ, but it separates the role of transaction costs from that of asset dynamics.

Chapter 2

No Arbitrage under Market Frictions

The First Fundamental Theorem of Asset Pricing states that an asset process X is essentially arbitrage-free if and only if there is a measure Q, equivalent to the physical measure P, such that X is a local martingale under Q.

The one-period version of this theorem goes back to the seminal work of Arrow and Debreu on elementary securities. In the last two decades, it has been progressively extended to multiperiod models with finite Ω by Harrison and Pliska [HP81], in discrete time by Dalang, Morton and Willinger [DMW90] and in full generality by Delbaen and Schachermayer [DS94, DS98].

With transaction costs, this result is no longer valid. While the existence of a martingale measure still remains a sufficient condition for the absence of arbitrage, it implies a total ban on non-semimartingale models, as the semimartingale property is preserved under the change to an equivalent measure.

In finite discrete time, Jouini and Kallal [JK95] have shown, in the context of simple strategies, that the absence of arbitrage in presence of transaction costs is equivalent to the existence of an auxiliary asset process, lying within the bid-ask spread, which admits a martingale measure. In the same setting, this result has been extended to general strategies by Kabanov and Stricker [KS01] in the case of finite Ω and by Kabanov, Rásonyi and Stricker [KRS01] for general Ω under the auxiliary assumption of "efficient friction". In fact, in presence of transaction costs it turns out that even in discrete time there are different notions of no arbitrage. Recently, Schachermayer [Sch01a] has shown that an equivalent condition for no arbitrage under transaction costs in finite discrete time is given by the "robust no-arbitrage" assumption.

In continuous time, the problem of characterizing arbitrage free models with transaction costs in terms of martingale conditions seems still open, but the literature on the discrete time case hints that simple sufficient conditions can be established.

In order to do this, we introduce the following:

Definition 2.1. *Given an adapted, strictly positive process $\gamma_t = (\gamma_t^1, \dots, \gamma_t^d)$, a process X is* γ-arbitrage free *if there exists a process $\tilde{X}$ and a probability Q equivalent to P such that $(1-\gamma_t^i)X_t^i \leq \tilde{X}_t^i \leq (1+\gamma_t^i)X_t^i$ almost surely in $dtdP$ for all i, and $\tilde{X}$ is a local martingale under Q.*

The main idea of this definition is that, in presence of transaction costs, an economically small (i.e. within the bid-ask spread) perturbation of the asset process X should not change the arbitrage properties of the model. Hence, perturbing a model which is arbitrage-free even without frictions (since it has a martingale measure), we should obtain a model which is arbitrage-free with transaction costs.

Of course, the arbitrage properties of a γ-arbitrage free model will depend on the relationship between γ and the transaction cost process k.

In this chapter we obtain the following no-arbitrage criterion:

Proposition 2.2. *If X is γ-arbitrage free, and $\gamma_t \leq k_t$ (i.e. $\gamma_t^i \leq k_t^i$ a.s. in $dtdP$ for all i) then X is arbitrage-free with transaction costs k.*

This result allows to consider non-semimartingale models within a no-arbitrage setting. In the next chapters, we shall also see that when γ is strictly smaller than k (in a sense to be made precise) the space of admissible strategies inherits some compactness properties similar to the frictionless case. By contrast, when $\gamma = k$ the interplay between transaction costs and arbitrage becomes more subtle, and compactness may be lost.

In this chapter, we shall denote the liquidation value of the portfolio as $V_t^{c,X,k}(\theta)$, to stress its the dependence on k. On the other hand, we will simply write $V_t^{c,X}$ for $k = (0, \dots, 0)$ (the case of a frictionless market).

2.1 Arbitrage with model uncertainty

Our investigation begins with an equivalence result, which states that an arbitrage with proportional transaction costs k is essentially equivalent to an arbitrage in a market without transaction costs, but with an uncertainty in the asset price not exceeding kX. Such equivalence will provide additional support, from a different point of view, to the choice of working with finite variation strategies.

Suppose now that transaction costs are not present, but that the asset process X_t is known only up to an error ε_t not exceeding $k_t X_t$ in absolute value. In this case, the portfolio value at time t will be given by:

$$V_t^{c,X+\varepsilon}(\theta) = c + (\theta \cdot (X+\varepsilon))_t$$

and its lower and upper bounds will be, respectively:

$$^{-}V_t^{c,X,k}(\theta) = \inf_{\tilde{X}\in\mathcal{V}} V_t^{c,\tilde{X}}(\theta) \quad \text{and} \quad ^{+}V_t^{c,X,k}(\theta) = \sup_{\tilde{X}\in\mathcal{V}} V_t^{c,\tilde{X}}(\theta)$$

where $\mathcal{V} = \{X+\varepsilon : \varepsilon_t \text{ adapted to } \mathcal{F}_t,\ |\varepsilon_t| \leq kX_t\}$.

Accordingly, we can give the definition of admissible strategy:

Definition 2.3. *An $\mathcal{F}_t$-predictable, X-integrable process $\theta = (\theta_1, \dots, \theta_d)$ is an* admissible strategy *with uncertainty k if there exists some $c > 0$ such that $^{-}V_t^{c,X,k}(\theta) \geq 0$ a.s. .*

It is natural to ask whether we can estimate the effect of the uncertainty ε on the portfolio value, that is the length of the interval $[^{-}V_t^{c,X,k}(\theta), ^{+}V_t^{c,X,k}(\theta)]$.

The next example shows that in general this is not the case:

Example 2.4. Consider Y_t, a reflected Brownian Motion between $-\delta$ and δ, and denote by $X_t = e^{Y_t}$. Of course, we have that $e^{-\delta} \leq X_t \leq e^{\delta}$, so that X can be made arbitrarily close to 1 with for small values of δ. Choosing $\theta_t = -X_t$, we have:

$$(\theta \cdot X)_t = -\int_0^t X_s dX_s = \frac{1}{2}\left(\langle X\rangle_t - X_t^2 + 1\right) \geq \frac{1}{2}\left(te^{-2\delta} - e^{2\delta} + 1\right)$$

and as $\delta \to 0$ the last term converges to t. On the other hand, for $\delta = 0$ (i.e. $X \equiv 1$), we obviously have $(\theta \cdot X)_t = 0$ for all θ. In other words, an arbitrarily small (with probability 1) uncertainty in the asset process X can lead to a large change in the portfolio value.

The next proposition shows that strategies of a.s. finite variation are the largest possible class for which the above phenomenon does not occur. The proof borrows from an idea of P.A. Meyer, often used to show that naïve stochastic integration is impossible (see for instance Protter [Pro90]). Recall that $X_t^* = \sup_{s\le t}|X_s|$.

Proposition 2.5. *Let θ be a predictable process. The following conditions are equivalent:*

i) There exists a positive constant M such that, for any adapted continuous process X,

$$|(\theta\cdot X)_t| \le MX_t^* \qquad a.s.$$

ii) θ has finite variation.

In addition, the smallest constant for which i) is satisfied for all X is $2|D\theta|_t$, while for X positive it is $|D\theta|$.

Proof. *ii)* $\Rightarrow$ *i)* Let $\theta^k = \sum_i \theta_{\tau_i} 1_{(\tau_i^k,\tau_{i+1}^k]}$ be a sequence of simple predictable processes converging to θ almost surely. We have:

$$(\theta^k\cdot X)_t = \Big(\sum_i \theta_{\tau_i}(X_{\tau_{i+1}} - X_{\tau_i})\Big)_t = \theta_t X_t - \sum_{\tau_i\le t} X_{\tau_{i+1}}(\theta_{\tau_{i+1}} - \theta_{\tau_i}) =$$
$$= \sum_{\tau_i\le t}(X_t - X_{\tau_{i+1}})(\theta_{\tau_{i+1}} - \theta_{\tau_i}) \le (X_t - X)_t^* \sum_{\tau_i\le t}|\theta_{\tau_{i+1}} - \theta_{\tau_i}| \le (X_t - X)_t^*|D\theta|_t$$

and passing to the limit as $k\to\infty$, we obtain that:

$$(\theta\cdot X)_t \le (X_t - X)_t^*|D\theta|_t \le 2X_t^*|D\theta|_t$$

If X is positive, we have that, for all $s\le t$:

$$-X_t^* \le -X_s \le X_t - X_s \le X_t \le X_t^*$$

and hence $(X_t - X)_t^* \le X_t^*$.

i) $\Rightarrow$ *ii)* For a given simple predictable process θ^k, define the linear operator $T_k : C([0,t]) \mapsto \mathbb{R}$ as follows:

$$T_k(X(\omega)) = (\theta^k \cdot X)_t(\omega) = \theta_t X_t - \sum_{\tau_i \leq t} X_{\tau_{i+1}}(\theta_{\tau_{i+1}} - \theta_{\tau_i})(\omega)$$

Choosing $\tilde{X}_t = 0$ and $\tilde{X}_{\tau_{i+1}} = -\mathrm{sgn}(\theta_{\tau_{i+1}} - \theta_{\tau_i})$, we have that:

$$\|T_k\| \geq T_k(\tilde{X}) = \sum_{\tau_i \leq t} |\theta_{\tau_{i+1}} - \theta_{\tau_i}|$$

and, passing to the limit, $\sup_k \|T_k\| \geq |D\theta|_t$. By assumption, $T_k(X(\omega)) \leq MX_t^*(\omega)$ for all k and hence $\sup_k T_k(X(\omega)) < \infty$. By the Banach-Steinhaus theorem it follows that $\sup_k \|T_k\| < \infty$. $\square$

In the context of model uncertainty, we define an arbitrage strategy in terms of its payoff in the worst case scenario:

Definition 2.6. *We say that an admissible strategy θ is an* arbitrage opportunity with uncertainty k *if* ${}^{-}V_t^{c,X,k}(\theta) \geq 0$ *a.s. and* $P({}^{-}V_t^{c,X,k}(\theta) > 0) > 0$.

We now show that Definitions 1.3 and 2.6 are equivalent:

Proposition 2.7. *For all admissible strategies θ, we have that:*

$${}^{-}V_t^{c,X,k}(\theta) = V_t^{c,X,k}(\theta)$$

Proof. For any θ and all $|\varepsilon_t| \leq k_t X_t$ we have that:

$$\begin{aligned}(\theta \cdot (X+\varepsilon))_t =& (\theta \cdot X)_t + \theta_t \varepsilon_t - \theta_{0^+}\varepsilon_0 - \int_{(0,t]} \varepsilon_s dD\theta_s = \\ =& (\theta \cdot X)_t + \theta_t \varepsilon_t - \int_{[0,t]} \varepsilon_s dD\theta_s \geq V_t^{c,X,k}(\theta)\end{aligned}$$

and hence ${}^{-}V_t^{c,X,k}(\theta) \geq V_t^{c,X,k}(\theta)$. In fact, equality can be obtained choosing $\varepsilon_s = \frac{dD\theta_s}{d|D\theta|_s} k_s X_s$ for $s < t$ and $\varepsilon_t = \frac{\theta_t}{|\theta_t|} k_t X_t$. $\square$

From the above proposition, we obtain the following:

Corollary 2.8. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \Omega, k)$ be a market model. The following conditions are equivalent:*

i) the market with transaction costs k allows arbitrage.

ii) the market with uncertainty k allows arbitrage.

2.2 No-arbitrage criteria under transaction costs

We now turn to sufficient conditions for no arbitrage:

Lemma 2.9. *Let $X, \tilde{X}$ be two processes such that:*

$$|X_t - \tilde{X}_t| \leq k_t X_t - \tilde{k}_t \tilde{X}_t \quad a.s. \tag{2.1}$$

Then, for all admissible θ, we have that:

$$V_t^{c,X,k}(\theta) \leq V_t^{c,\tilde{X},\tilde{k}}(\theta) \quad a.s.$$

Proof. We have:

$$V_t^{c,X,k}(\theta) = V_t^{c,\tilde{X},\tilde{k}}(\theta) + \tag{2.2}$$

$$+(\theta \cdot (X - \tilde{X}))_t - \int_{[0,t]} (k_s X_s - \tilde{k}_s \tilde{X}_s) d|D\theta|_s - |\theta|_t (k_t X_t - \tilde{k}_t \tilde{X}_t) \tag{2.3}$$

Denoting by $\varepsilon_t = (X - \tilde{X})_t$ and recalling that:

$$(\theta \cdot (X - \tilde{X}))_t = \theta_t \varepsilon_t - \int_{[0,t]} \varepsilon_s dD\theta_s$$

by the assumption (2.1) we obtain that (2.3) is nonpositive. □

Using the case of $\tilde{k} = 0$ in the above Lemma and the characterization of arbitrage-free markets in terms of martingale measures, the following no-arbitrage criterion follows:

Corollary 2.10. *If there exists a process $\tilde{X}$ and a probability $\tilde{P}$ such that:*

i) $\tilde{P}$ is equivalent to P and $\tilde{X}$ is a $\tilde{P}$-local martingale with respect to $\mathcal{F}_t$ ($\tilde{P}$ is a local martingale measure for $\tilde{X}$).

ii) $|X^i - \tilde{X}^i| \leq k_t^i X^i$ a.s. for all i.

then $(X, \mathcal{F}_t, k)$ is arbitrage free.

Proof. We need to show that if $V_t^{0,X,k}(\theta) \geq 0$ almost surely, and θ is admissible, then $V_t^{0,X,k}(\theta)$ is identically 0.

By Lemma 2.9, an arbitrage strategy for X under transaction costs is an arbitrage strategy for $\tilde{X}$ without transaction costs. But $\tilde{X}$ has the local martingale measure Q, therefore it is arbitrage free by Fundamental Theorem of Asset Pricing (see for instance [DS94] Theorem 1.1), and $V_t^{0,X,k}(\theta)$ can only be 0. □

Another application of Lemma 2.9 is given by the following criterion, which says that, roughly speaking, X is arbitrage-free with transaction costs k if we can find an arbitrage-free process $\tilde{X}$ with transaction costs $\tilde{k} < k$ sufficiently close to X.

Proposition 2.11. *Let X and $\tilde{X}$ two processes and $k, \tilde{k}, \varepsilon > 0$ such that:*

$$|X^i - \tilde{X}^i| \leq \min(\varepsilon^i, k^i - \tilde{k}^i(1+\varepsilon^i))X \quad a.s.$$

If $(\tilde{X}, \mathcal{F}_t, \tilde{k})$ is arbitrage-free, then also $(X, \mathcal{F}_t, k)$ is arbitrage-free.

Proof. We have the inequality:

$$kX - \tilde{k}\tilde{X} = (k - \tilde{k})X - \tilde{k}(X - \tilde{X}) \geq (k - \tilde{k}(1-\varepsilon))X \geq |X - \tilde{X}|$$

which implies that the assumption of Lemma 2.9 is satisfied. It follows that an arbitrage strategy in X would be dominated by a strategy in $\tilde{X}$, which is absurd. □

Remark 2.12. Corollary 2.10 provides a no-arbitrage criterion for the process $\tilde{X}$, provided that we can find a process $\tilde{X}$ which lies within the bid-ask spread of X, and is a local martingale under the equivalent measure Q and the filtration $\mathcal{F}_t$.

It seems natural to ask if this condition can be relaxed, requiring $\tilde{X}$ to be a local martingale only with respect to its natural filtration, but the next example shows that this is not the case:

Example 2.13. Consider the following discrete-time binomial model:

$$\tilde{Y}_{2n} = \tilde{Y}_{2n+1} = cn + \sum_{i=1}^{n} \xi_i$$

where $\{\xi_i\}_{i\in\mathbb{N}}$ is a sequence of IID random variables, such that $P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$. Define Y as follows:

$$Y_n = \begin{cases} \tilde{Y}_n & \text{for } n \text{ even} \\ \tilde{Y}_n + \lambda(\tilde{Y}_{n+1} - \tilde{Y}_n) & \text{for } n \text{ odd} \end{cases}$$

and denote by $X_n = e^{Y_n}$ and $\tilde{X}_n = e^{\tilde{Y}_n}$. For a suitable choice of c $\tilde{X}$ is a martingale under its natural filtration, and therefore it is clearly arbitrage-free. Also, we always have that $|\tilde{Y}_n - Y_n| \leq \lambda$, therefore for a sufficiently small value of λ we have that $|\tilde{X}_n - X_n| \leq kX_n$, for any $k > 0$.

On the other hand, for odd n we have that:

$$\Delta Y_n = \lambda(\tilde{Y}_{n+1} - \tilde{Y}_n) = \lambda \Delta Y_{n+1}$$

And therefore at any such step the next increment of Y (and hence X) is known. This allows an obvious arbitrage, given by the strategy $\xi_n = 1_{\{n \text{ odd}, \ \Delta Y_n > 0\}}$. In fact, for a small value of λ transaction costs become negligible with respect to the gain at the next step.

2.3 Notes

Remark 2.14. Kabanov, Rásonyi and Stricker [KRS01] do not formulate their equivalent condition for no-arbitrage in terms of the existence of an auxiliary asset $\tilde{X}$, but in terms of dual processes with values in the dual of the solvency cone K^*. However, as observed in Kabanov and Stricker [KS01] translating their condition to the setting of bid-ask spreads, it boils down to Definition 2.3.

Remark 2.15. Since in presence of transaction costs there are various kinds of arbitrage, and correspondingly various no-arbitrage properties, it is worthwhile to investigate under which conditions on γ and k these conditions are satisfied.

In fact, it can be checked that:

Proposition 2.16. *Let* $X = \{X_i\}_{0 \le i \le n}$ *be adapted, strictly positive discrete time process. If* X *is* γ*-arbitrage free, we have that:*

i) if $\gamma^i_n \le k^i_n$ *a.s. for all* i, n*, then* X *satisfies the* $\mathbf{NA}^s$ *condition in [KRS01];*

ii) if $\gamma^i_n < k^i_n$ *a.s. for all* i, n*, then* X *satisfies the* **NA** *condition in [KRS01];*

iii) if $\min_{1 \le i \le d} \operatorname{essinf}_{0 \le n \le T} (k^i_n - \gamma^i_n) > 0$ *a.s., then* X *satisfies the "robust no-arbitrage" condition in [Sch01a].*

In chapter 5 we will use condition *iii*) above in order to study optimization problems with budget constraints, as it implies for the set of admissible strategies $\mathcal{A}^k_c$ compactness properties analogous to those in the frictionless case. We will also show with a counterexample that these properties may be lost under the weaker condition *i*).

Chapter 3

Semimartingale Problems with Integrability Conditions

In this chapter we consider optimization problems on a fixed horizon, defining the spaces of admissible strategies in terms of integrability conditions on the gain process.

As mentioned in chapter 1, this is the most natural approach when the objective function itself implies integrability, such in the case of *mean-variance hedging*. However, here we shall consider only convex and decreasing risk functionals, which obviously exclude that approach. This choice is motivated by consistency considerations, discussed in chapter 4, which show that the mean-variance criterion leads to systematic dissipation of wealth in presence of transaction costs.

Here we consider the slightly more general case of an agent facing some contingent liability $H = (H_X, H_B)$ at time T, which requires the payment of H_X shares of the risky asset, and H_B units of the numeraire. Essentially, this amounts to considering contingent claims settled both in cash and stock.

We separate the *market value* (as opposed to the *liquidation value*) of a portfolio in the initial capital, the the trading gain, and the transaction cost, namely:

$$V_t^c(\theta) = c + G_t(\theta) - C_t(\theta)$$

At the terminal date T, the payment of the liability H and the liquidation of the remaining portfolio will give a payoff equal to:

$$V_T^c(\theta) - k_T X_T |\theta_T - H_X| - X_T H_X - H_B$$

which must be evaluated according to the agent preferences.

3.1 Spaces of Strategies

In the literature on markets with incomplete information, the following spaces of strategies are often considered, especially for $p = 2$:

$$\Theta^p = \{\theta : \theta\ \mathcal{F}_t\text{-predictable},\ G_T(\theta) \in L^p(P)\}$$

The space Θ^p can be endowed with the topology induced by the map $G_T : \theta \mapsto \int_{[0,T]} \theta_t dX_t$. It is clear that $G_T(\Theta^p)$ is a linear subspace of L^p: if X is a continuous martingale, it turns out that it is also closed, as it follows from Theorems A.3 and A.4, in the cases $p > 1$ and $p = 1$ respectively.

The presence of transaction costs in fact forces a much narrower set of admissible strategies than Θ^p. As we argued in the first chapter, in a market with proportional costs we should only consider strategies with finite variation. This leads us to define the following spaces:

$$\Theta_C^p = \{\theta \in \Theta^p,\ C_T(\theta) \in L^p(P)\}$$

endowed with the norm:

$$\|\cdot\|_{\Theta_C^p} :\ \theta \to \left(\left\| \int_0^T \theta_t dX_t \right\|_p^p + \left\| \int_{[0,T]} k_t X_t d|D\theta|_t \right\|_p^p \right)^{\frac{1}{p}} =$$
$$= \left(\|G_T(\theta)\|_p^p + \|C_T(\theta)\|_p^p \right)^{\frac{1}{p}}$$

We begin our discussion with the following result:

Proposition 3.1. *Let X be a continuous local martingale, and k a continuous, adapted process, such that $\tilde{k} = \min_{t\in[0,T]} k_t X_t > 0$ for a.e. ω. Then Θ_C^p is a Banach space for all $p \geq 1$.*

Proof. Θ_C^p is a linear subspace of Θ^p, therefore $\|G_T(\theta)\|_p$ is a norm. Hence it is sufficient to prove that $\|C_T(\theta)\|_p$ is also a norm, and that the space is complete. Trivially, $\|C_T(\lambda\theta)\|_p = |\lambda| \, \|C_T(\theta)\|_p$. Let $\theta, \eta \in \Theta_C^p$. We have:

$$\|C_T(\theta+\eta)\|_p = \left\| \int_{[0,T]} k_t X_t d|D(\theta+\eta)|_t \right\|_p \leq$$
$$\leq \left\| \int_{[0,T]} k_t X_t d|D\theta|_t + \int_{[0,T]} k_t X_t d|D\eta|_t \right\|_p \leq \|C_T(\theta)\|_p + \|C_T(\eta)\|_p$$

It remains to show that Θ_C^p is complete. Let θ^n be a Cauchy sequence for Θ_C^p: since it is also Cauchy in Θ^p and X is continuous, it follows that Θ^p is complete by Theorems A.3 and A.4, and we can assume that $\theta^n \to \theta$ in Θ^p.

We now show that convergence holds in the $\|C_T(\theta)\|_p$ norm. Through a standard Borel-Cantelli argument (see for instance Shiriayev [Shi84], page 257), we obtain a strategy θ' and a subsequence n_k such that $C_T(\theta^{n_k}-\theta') \to 0$ almost surely. Since θ^n is a Cauchy sequence in $\|C_T(\theta)\|_p$, by the Fatou's Lemma we have

$$E\left[C_T(\theta^m - \theta')^p\right] = E\left[\liminf_{k\to\infty} C_T(\theta^m - \theta^{n_k})^p\right] \leq \liminf_{k\to\infty} E\left[C_T(\theta^m - \theta^{n_k})^p\right] < \varepsilon$$

which provides the desired convergence. □

Remark 3.2. Θ_C^p is not a Hilbert space even for $p = 2$. In fact, it is easily checked that the equality $|\theta+\eta|^2 + |\theta-\eta|^2 = 2|\theta|^2 + 2|\eta|^2$, which is valid in any Hilbert space, is not satisfied from the deterministic strategies $\theta_t = 1_{\{t<\frac{1}{3}T\}}$ and $\eta_t = 1_{\{t\geq\frac{2}{3}T\}}$.

Remark 3.3. Θ_C^p is generally not *separable*. To see this, observe that the set of deterministic strategies $\{\theta^x\}_{x\in[0,T]}$, where $\theta_t^x = 1_{\{t\geq x\}}$, is uncountable, and $\|\theta^x - \theta^y\|_{\Theta_C^p} \geq k_x(\omega) + k_y(\omega)$ for all $x \neq y$ and for all $p \geq 1$. If $k_t X_t$ is uniformly bounded away from zero, it follows that $\|\theta^x - \theta^y\|_{\Theta_C^p} \geq c$ for some positive c, which proves the claim.

The following inequality states a continuous immersion of Θ_C^r into Θ^p, for $r > p$, provided that $\frac{\langle X\rangle_T^{\frac{1}{2}}}{\tilde{k}}$ is sufficiently integrable.

Proposition 3.4. *Let X be a continuous local martingale. For any $p, q \geq 1$, we have:*

$$\|G_T(\theta)\|_p \leq \|C_T(\theta)\|_{pq} \left\| \frac{\langle X \rangle_T^{\frac{1}{2}}}{\tilde{k}} \right\|_{pq'}$$

where $q' = \frac{q}{q-1}$.

Proof. For any p, we have, by the Burkholder-Davis-Gundy inequality:

$$\begin{aligned} E\left[|G_T(\theta)|^p\right] \leq E\left[\left(\int_0^T \theta_t^2 d\langle X \rangle_t\right)^{\frac{p}{2}}\right] \leq E\left[\left(\sup_{t \leq T} |\theta_t|\right)^p \langle X \rangle_T^{\frac{p}{2}}\right] \leq \\ \leq E\left[|D\theta|([0,T])^p \langle X \rangle_T^{\frac{p}{2}}\right] \leq E\left[C_T(\theta)^p \left(\frac{\langle X \rangle_T^{\frac{1}{2}}}{\tilde{k}(\omega)}\right)^p\right] \leq \\ \leq E\left[C_T(\theta)^{pq}\right]^{\frac{1}{q}} E\left[\left(\frac{\sqrt{\langle X \rangle_T}}{\tilde{k}(\omega)}\right)^{pq'}\right]^{\frac{1}{q'}} \end{aligned}$$

and, raising both sides to the power $\frac{1}{p}$, the thesis follows. □

Remark 3.5. Proposition 3.4 admits a simple financial interpretation. The transaction cost needed for a gain with a high moment of order p is bounded if the asset itself has a sufficiently high moment of the same order (a buy and hold strategy does the job). Otherwise, the strategy itself must amplify the swings of the market. In this case, the less the market is volatile, the higher the moment of the strategy.

Remark 3.6. Denoting the $\mathcal{H}^p$ norm of a martingale by $\|M\|_{\mathcal{H}^p} = E\left[\langle X \rangle_T^{\frac{p}{2}}\right]$, it is clear from the proof of Proposition 3.4 that we also have:

$$\|G_T(\theta)|_{\mathcal{H}^p} \leq \|C_T(\theta)\|_{pq} \left\| \frac{\langle X \rangle_T^{\frac{1}{2}}}{\tilde{k}} \right\|_{pq'}$$

This is trivial for $p > 1$, as the $\mathcal{H}^p$ norm is equivalent to the L^p norm. On the other hand, the space $\mathcal{H}^1$ is strictly smaller than L^1. In this case, the last observation states that the gain $G_T(\theta)$ belongs to $\mathcal{H}^1$ and, *a fortiori*, is a uniformly integrable martingale.

Remark 3.3 suggests that the norm topology in Θ_C^p is too restrictive to provide sufficient compactness on the space of strategies. The following lemma provides a more reasonable alternative:

Lemma 3.7. *Let X be a continuous local martingale, and $G_T(\theta^n) \to G_T(\theta)$ in the L^p-norm. Then:*

i) if $p > 1$, up to a subsequence $\theta^n \to \theta$ a.s. in $d\langle X\rangle_t dP$;

ii) if $p = 1$, there exists some η^n, convex combinations of stoppings of θ^n, such that $\eta^n \to \theta$ a.s. in $d\langle X\rangle_t dP$.

Proof. *i)* Let τ_h be a reducing sequence of stopping times for the local martingale X_t. For any h, the Burkholder-Davis-Gundy inequality yields:

$$E\left[|G_{T\wedge\tau_h}(\theta^n) - G_{T\wedge\tau_h}(\theta)|^p\right] \geq c_p E\left[\left|\int_0^{T\wedge\tau_h} (\theta_t^n - \theta_t)^2 d\langle X\rangle_t\right|^{\frac{p}{2}}\right] \tag{3.1}$$

for some positive constant c_p. Since the left-hand side converges to zero, it follows that $\int_0^{T\wedge\tau_h}(\theta_t^n - \theta_t)^2 d\langle X\rangle_t$ also converges to zero in probability, and $\theta^n \to \theta$ in the measure $d\langle X\rangle_t dP$. Up to a subsequence, convergence holds a.s., and since $d\langle X^{\tau_h}\rangle_t dP$ is a sequence of measures increasing to $d\langle X\rangle_t dP$, we conclude that $\theta^n \to \theta$ a.s. in $d\langle X\rangle_t dP$.

ii) The situation is more delicate here, because (3.1) is not true in L^1. Denote by $\tau_h = \inf\{t : |G_t(\theta)| \geq h\}$. The stopped martingales $G_{t\wedge\tau_h}(\theta)$ clearly converge to $G_t(\theta)$ almost surely. For each h, $G_{t\wedge\tau_h}(\theta) \in \mathcal{H}^1$ and we can apply Corollary A.7, obtaining that for some stopping times $T_{n,h}$, $G_{t\wedge T_{n,h}}(\theta^n) \rightharpoonup G_{t\wedge\tau_h}(\theta)$ in $\sigma(\mathcal{H}^1, BMO)$. Up to a sequence of convex combinations $\xi_t^{n,h} = \sum_{j=n}^{M_n} \beta_j^{n,h} G_{t\wedge T_{j,h}}(\theta^j)$, we can assume that $\xi_t^{n,h} \to G_{t\wedge\tau_h}(\theta)$ in the strong topology of $\mathcal{H}^1$. Observe also that $\xi_t^{n,h} = G_t(\eta^{n,h})$, where $\eta_t^{n,h} = \sum_{j=n}^{M_n} \beta_j^{n,h} \theta_t^j 1_{\{t<T_{j,h}\}}$, and that if $h' < h$, then $(\xi^{n,h})^{\tau_{h'}} \to G_{t\wedge\tau_{h'}}(\theta)$. Hence, by a diagonalization argument, we consider the sequence $\eta^{n,n}$, which satisfies the condition $G_{t\wedge\tau_h}(\eta^{n,n}) \to G_{t\wedge\tau_h}(\theta)$ in the $\mathcal{H}^1$ norm for all h.

In other words:

$$\lim_{n\to\infty} E\left[\left(\int_0^{T\wedge\tau_h} (\eta_t^{n,n} - \theta_t)^2 d\langle X\rangle_t\right)^{\frac{1}{2}}\right] = 0$$

This means that $\eta^{n,n} \to \theta$ in the measure $d\langle X^{\tau_h}\rangle_t dP$, and up to a subsequence, a.s. As in i), we conclude that $\eta^{n,n} \to \theta$ a.s. in $d\langle X\rangle_t dP$. □

The next Proposition provides the lower semicontinuity of the cost process, with respect to the convergence in $dtdP$. Intuitively, this means that taking limits can only reduce transaction costs, because in the limit strategy some transactions may cancel out, while no new ones can arise.

Proposition 3.8. *If θ^n is bounded in Θ^p_C, and $\theta^n_t \to \theta_t$ a.s. in $dtdP$ then:*

$$C_T(\theta) \leq \liminf_{n\to\infty} C_T(\theta^n) \qquad \textit{for a.e. } \omega \tag{3.2}$$

and

$$\|C_T(\theta)\|_p \leq \liminf_{n\to\infty} \|C_T(\theta^n)\|_p \tag{3.3}$$

for all $p \geq 1$.

The proof requires a few lemmas:

Lemma 3.9. *For a fixed ω, let $\theta^n(\omega)_t \to \theta(\omega)_t$ for a.e. t, and $|D\theta(\omega)^n|([0,T]) < C$ uniformly in n. Then $D\theta(\omega)^n \rightharpoonup D\theta(\omega)$ in the weak star topology of Radon measures.*

Proof. For all $\phi \in C^\infty_c[0,T]$, we have:

$$\nu(\phi) = \lim_{n\to\infty} \int_{[0,T]} \phi_t dD\theta^n_t = -\lim_{n\to\infty} \int_{[0,T]} \phi'_t \theta^n_t dt = -\int_{[0,T]} \phi'_t \theta_t dt = D\theta(\phi)$$

It remains to show that the distribution $D\theta$ is in fact a Radon measure, and this follows from the inequality:

$$D\theta(\phi) \leq \sup_{t\in[0,T]} |\phi(t)| \limsup_{n\to\infty} |D\theta^n|([0,T]) \leq C \sup_{t\in[0,T]} |\phi(t)|$$

which completes the proof. □

The following is a standard result in measure theory (see for instance [AFP99]):

Lemma 3.10. *Let $\mu^n \rightharpoonup \mu$, where μ^n, μ are Radon measures on I, and convergence is meant in the weak star sense. Then $|\mu| \leq \liminf_{n\to\infty} |\mu^n|$.*

Proof. of Proposition 3.8 By assumption, for a.e. ω, $\theta^n_t(\omega) \to \theta_t(\omega)$ for a.e. t. To prove (3.2), we show that for all subsequences n_j for which $C_T(\theta^{n_j}(\omega))$ converges, we have:

$$C_T(\theta(\omega)) \leq \lim_{j\to\infty} C_T(\theta^{n_j}(\omega)) \tag{3.4}$$

If $C_T(\theta^{n_j}(\omega)) \to \infty$, then (3.4) is trivial. If not, then $C_T(\theta^{n_j}(\omega)) < M(\omega)$ for all j and hence

$$|D\theta^{n_j}(\omega)|([0,T]) < \frac{M(\omega)}{\tilde{k}(\omega)}$$

Lemma 3.9 implies that $D\theta^{n_j}(\omega) \rightharpoonup D\theta(\omega)$. By Lemma 3.10, we obtain:

$$C_T(\theta(\omega)) = \int_{[0,T]} k_t X_t d|D\theta(\omega)| \leq \lim_{j\to\infty} \int_{[0,T]} k_t X_t d|D\theta^{n_j}(\omega)| = \lim_{j\to\infty} C_T(\theta^{n_j}(\omega))$$

and (3.2) follows. For (3.3), notice that:

$$\|C_T(\theta(\omega))\|_p^p = E\left[\left(\int_{[0,T]} k_t X_t d|D\theta(\omega)|\right)^p\right] \leq E\left[\liminf_{n\to\infty}\left(\int_{[0,T]} k_t X_t d|D\theta^n(\omega)|\right)^p\right] \leq$$
$$\leq \liminf_{n\to\infty} E\left[\left(\int_{[0,T]} k_t X_t d|D\theta^n(\omega)|\right)^p\right] = \liminf_{n\to\infty} \|C_T(\theta^n(\omega))\|_p^p < \infty$$

where the last inequality follows from the uniform boundedness of θ^n in Θ^p_C, and the previous one holds by Fatou's Lemma. □

3.2 Existence of Optimal Strategies

This section contains the main existence results for optimal hedging strategies in unconstrained incomplete markets with proportional transaction costs, in the local martingale case.

In general, the existence of a minimum requires two basic ingredients: relative compactness of minimizing sequences (up to some transformation which leaves them minimizing), and lower semicontinuity of the functional.

Compactness is obviously much easier in L^p spaces with $p > 1$, since it coincides with boundedness, but this kind of information is rare to obtain in applications. On the other hand, some measures of risk (maximization of utility and minimization of shortfall) seem to provide natural bounds on the L^1 norms of optimizing portfolios at expiration.

Moreover, in the next section we shall see that if X is a semimartingale, then a minimization problem can be reduced through a change of measure to a problem in L^1. This shows that the L^1 case is both mathematically more challenging, and the most relevant in applications.

We also need the risk functional $F : \theta \mapsto \rho(V_T^c(\theta) - H)$ to be lower semicontinuous (shortly l.s.c.). Proposition 3.8 shows that in general $V_T^c(\cdot)$ is upper semicontinuous with respect to a.s. convergence in $dtdP$, but not necessarily continuous. This means that we need a decreasing ρ to ensure the semicontinuity of F. Also, we are going to take convex combinations of minimizing strategies, and we need a convex ρ to leave them minimizing.

This leads to the following definition, already introduced in Chapter 1:

Definition 3.11. *We define a* convex decreasing risk functional *as a function $\rho : L^p \mapsto \mathbb{R} \cup \{+\infty\}$, satisfying the following properties:*

i) ρ is convex;

ii) if $X(\omega) \leq Y(\omega)$ for a.e. ω, then $\rho(X) \geq \rho(Y)$ (ρ is decreasing);

iii) ρ has the Fatou property. Namely, if $X_n \to X$ a.s., then

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$$

We easily see that the above definition provides the desired properties of semicontinuity and convexity:

Lemma 3.12. *Let ρ be a convex decreasing functional, and $c > 0$. Denoting by $H(\theta) = k_T X_T |\theta_T - H_X| + X_T H_X + H_B$ and $F : \theta \mapsto \rho\left(V_T^c(\theta) - H(\theta)\right)$, if $\theta^n \to \theta$ a.s. in $dtdP$, we have:*

i) F is convex;

ii) F is l.s.c. with respect to a.s. convergence in $dtdP$.

Proof. *i*) Since ρ is convex decreasing, and $V_T^c - H$ is concave, it follows that $F = \rho \circ (V_T^c - H)$ is convex.

ii) By Proposition 3.8, we have:

$$V_T^c(\theta) \geq \limsup_{n\to\infty} V_T^c(\theta^n)$$

Since ρ is decreasing, and $H(\theta)$ is continuous by definition:

$$\rho(V_T^c(\theta) - H(\theta)) \leq \rho\big(\limsup_{n\to\infty}(V_T^c(\theta^n) - H(\theta^n))\big)$$

and finally, by the Fatou property of ρ:

$$\rho(V_T^c(\theta) - H(\theta)) \leq \liminf_{n\to\infty} \rho(V_T^c(\theta^n) - H(\theta^n))$$

□

For convex decreasing functionals we are going to prove an existence result on bounded sets of Θ_C^p. Examples include σ-additive *coherent risk measures* (see Delbaen [Del00] for details).

A special class of these functionals consists of those which can be written as $\rho(X) = E[\nu(X)]$, where $\nu : \mathbb{R} \mapsto \mathbb{R}$ is a convex decreasing function. In this case, we show that an optimal strategy exists in the whole space Θ_C^1, since minimizing sequences are automatically bounded. Both the problems of *shortfall* minimization and *utility* maximization belong to this class.

Throughout this section, we make the following:

Assumption 3.13. *The measures $d\langle X\rangle_t dP$ and $dtdP$ are equivalent.*

This assumption implies that X cannot have intervals of constancy, and that its bracket process $\langle X\rangle_t$ cannot exhibit a Cantor-ladder behavior. It is necessary to draw inference on $C_T(\theta^n)$, which depends on convergence with respect to the measure $dtdP$, from the convergence of $G_T(\theta^n)$, which provides information in the measure $d\langle X\rangle_t dP$. In practice, it is satisfied by all diffusion models, even with Hölder coefficients or volatility jumps.

We start with risk minimization in Θ_C^p, with $p > 1$. In this case, we prove the existence of optimal strategies among those with a moment of order p not exceeding M. As a result, the minimum will generally depend on the particular bound considered.

The next lemma provides a class of weakly compact sets:

Lemma 3.14. *For $C \in \mathbb{R}^+$ and $p > 1$, the set*

$$B_{C,D} = \{\theta : \|G_T(\theta)\|_p \leq C, \|C_T(\theta)\|_p \leq D\}$$

is Θ^p-weakly compact for $D \in \mathbb{R}^+ \cup \{+\infty\}$.

Proof. Let $\theta^n \in B_{C,D}$. For $p > 1$, Θ^p is a reflexive Banach space (it is isometric to a closed subspace of L^p, which is reflexive). Hence, the set

$$B_C = \{\theta : \|G_T(\theta)\|_p \leq C\}$$

is weakly compact in Θ^p, and up to a subsequence $G_T(\theta^n) \rightharpoonup G_T(\theta) \in B_C$. Since $B_{C,D}$ is convex, by Theorem A.1 there exists a sequence $\eta^n \in B_{C,D}$ of convex combinations of θ^n, such that $G_T(\eta^n) \to G_T(\theta)$ in L^p. By Lemma 3.7, it follows that, up to a subsequence, $\eta^n \to \theta$ a.s. in $d\langle X\rangle_t dP$, and by Lemma 3.8, we conclude that $\theta \in B_{C,D}$. $\square$

Proposition 3.15. *Let ρ be a convex decreasing functional, $c > 0$ and $(H_B + X_T H_X, k_T X_T H_X) \in L^p(\Omega, \mathcal{F}_T, P)$, with $p > 1$. For any $M > 0$ let us denote*

$$\Theta^p_{C,M} = \{\theta \in \Theta^p_C,\ \|G_T(\theta)\|_p \leq M\}$$

Then the following minimum problem admits a solution:

$$\min_{\theta\in\Theta^p_{C,M}} \rho\left(V^c_T(\theta) - H(\theta)\right)$$

Proof. Let θ^n be a minimizing sequence, so that $F(\theta^n) \to \inf_{\theta\in\Theta^p_{C,M}} F(\theta)$. Since $\Theta^p_{C,M}$ is weakly compact by Lemma 3.14, up to a subsequence we can assume that $\theta^n \rightharpoonup \theta \in \Theta^p_{C,M}$. Then, by Theorem A.1, there exists a sequence of convex combinations $\eta^n = \sum_{k=n}^{\infty} \alpha^n_k \theta^k$, such that $\eta^n \to \theta$ in the strong topology. By Lemma 3.7, we can assume up to a subsequence that $\eta^n \to \theta$ in the $d\langle X\rangle_t dP$-a.s. convergence, and hence $dtdP$-a.s. by Assumption 3.13. Jensen's inequality implies that:

$$F(\eta^n) \leq \sum_{k=n}^{\infty} \alpha^n_k F(\theta^k) \leq \max_{n\leq k} F(\theta^k)$$

Passing to the limit:

$$\lim_{n\to\infty} F(\eta^n) \leq \lim_{n\to\infty} \max_{n\leq k} F(\theta^k) = \lim_{n\to\infty} F(\theta^n)$$

Finally, by the semicontinuity of F, we obtain:

$$F(\theta) \leq \lim_{n\to\infty} F(\theta^n)$$

hence θ is a minimizer. $\square$

We now turn to risk minimization in Θ^1_C. As mentioned before, this case has the advantage that minimizing sequences are bounded for some problems considered in applications. On the other hand, a few mathematical issues arise: a bounded sequence in L^1 does not necessarily converge, even in a weak sense, and the L^1 norm of a uniformly integrable martingale is not equivalent to the $\mathcal{H}^1$ norm.

It turns out that the first problem can be overcome through a result of Komlós [Kom67] (see also Schwartz [Sch86], for a shorter proof). We can circumvent the latter at the price of using stopping as a further transformation on minimizing sequences, besides extracting subsequences and taking convex combinations.

We start with the existence result for general convex decreasing risks:

Proposition 3.16. *Let ρ be a convex decreasing functional, $c > 0$ and $(H_B + X_T H_X, k_T X_T H_X) \in L^1(\Omega, \mathcal{F}_T, P)$. For any $M > 0$ the following minimum problem admits a solution:*

$$\min_{\theta \in \Theta^1_{C,M}} \rho\left(V^c_T(\theta) - H(\theta)\right)$$

Proof. Let θ^n be a minimizing sequence. By Komlós' Theorem (A.2), up to a subsequence of convex combinations $\eta^n = \sum_{k=n}^{M_n} \alpha^n_k \theta^k$ we can assume that $G_T(\eta^n) \to \gamma$ a.s. and in L^1, and by Yor's Theorem A.4, there exists some $\theta \in \Theta^1$ such that $\gamma = G_T(\theta)$. To see that $\theta \in \Theta^1_C$, first we apply Lemma 3.7, to obtain a sequence $\zeta^n = \sum_{j=n}^{M_n} \beta^n_j (\eta^j)^{T_{j,n}}$, such that $\zeta^n \to \theta$ a.s. Then Lemma 3.8 implies that $C_T(\theta) \in L^1(P)$, as required. By Jensen's inequality, we have:

$$F(\zeta^n) \leq \sum_{j=n}^{M_n} \beta^n_j (\eta^j)^{T_{j,n}} \leq \max_{n \leq j} F((\eta^j)^{T_{j,n}})$$

$$F(\eta^n) \leq \sum_{k=n}^{M_n} \alpha^n_k F(\theta^k) \leq \max_{n \leq k} F(\theta^k)$$

and, passing to the limit:

$$\lim_{n\to\infty} F(\zeta^n) \leq \lim_{n\to\infty} \max_{n \leq j} F((\eta^j)^{T_{j,n}}) = \lim_{n\to\infty} F(\eta^n)$$

$$\lim_{n\to\infty} F(\eta^n) \leq \lim_{n\to\infty} \max_{n \leq k} F(\theta^k) = \lim_{n\to\infty} F(\theta^n)$$

Finally, by the semicontinuity of F, we obtain:

$$F(\theta) \leq \lim_{n\to\infty} F(\theta^n)$$

hence θ is a minimum.

□

In the special case of ρ being the expectation of a convex decreasing function ν, it turns out that the optimal strategy in Θ^1_C coincides with that in $\Theta^1_{C,M}$, for some value of M. This is shown in the following

Proposition 3.17. *Let $\nu : \mathbb{R} \mapsto \mathbb{R}$ a strictly convex (at least in one point) decreasing function, $c > 0$ and $(H_B + X_T H_X, k_T X_T H_X) \in L^1(\Omega, \mathcal{F}_T, P)$. Then the problem*

$$\min_{\theta\in\Theta^1_C} E\left[\nu\left(V^c_T(\theta) - H(\theta)\right)\right]$$

admits a solution.

Proof. Let θ^n be a minimizing sequence, so that $F(\theta^n) \to \inf_{\theta\in\Theta^1_C} F(\theta)$. Since ν is strictly convex, we have that:

$$\nu(x) \geq a + bx^- - (b-\varepsilon)x^+ \tag{3.5}$$

which implies

$$E\left[Y^+\right] \leq \frac{1}{\varepsilon}\left(E\left[\nu(Y)\right] - a + bE\left[Y\right]\right)$$

for any integrable random variable Y. Substituting $Y = V^c_T(\theta^n) - H(\theta^n)$, we get:

$$E\left[(V^c_T(\theta^n) - H(\theta^n))^+\right] \leq \frac{1}{\varepsilon}\left(E\left[\nu(V^c_T(\theta^n) - H(\theta^n))\right] - a + bE\left[V^c_T(\theta^n) - H(\theta^n)\right]\right)$$

The first term in the right-hand side is bounded by assumption, since θ^n is a minimizing sequence. The second term is also bounded, because $E\left[V^c_T(\theta^n)\right] = c - E\left[C_T(\theta^n)\right] \leq c$ and $H(\theta^n)$ is integrable. Therefore $E\left[(V^c_T(\theta^n) - H(\theta^n))^+\right]$ is bounded and the inequality

$$E\left[V^c_T(\theta^n)^+\right] \leq E\left[(V^c_T(\theta^n) - H(\theta^n))^+\right] + E\left[H(\theta^n)^+\right]$$

implies that $E\left[V^c_T(\theta^n)^+\right]$ is bounded. In a similar fashion, (3.5) yields:

$$E\left[Y^-\right] \leq \frac{1}{b}(E\left[\nu(Y)\right] + (b-\varepsilon)E\left[Y^+\right] - a)$$

Substituting again $Y = V_T^c(\theta^n) - H(\theta^n)$, and using the foregoing result, we conclude that $\sup_n E\,|V_T^c(\theta^n)| < \infty$. Also:

$$E\,|V_T^c(\theta^n)| \geq E\,[-V_T^c(\theta^n)] = -c + E\,[C_T(\theta^n)]$$

which implies that $\sup_n E\,[C_T(\theta^n)] < \infty$, and hence $\sup_n E\,[|G_T(\theta^n)|] < \infty$. Now that boundedness is shown, the thesis follows from Proposition 3.16 for a suitable M. $\square$

Example 3.18 *(Shortfall risk).* In Proposition 3.17, choosing $\nu(x) = x^-$, we obtain the existence of a *shortfall minimizing* strategy, that is a solution of the problem

$$\max_{\theta \in \Theta_C^1} E\left[(H(\theta) - V_T^c(\theta))^+\right]$$

Without transaction costs, this problem has been solved for European options by Cvitanic and Karatzas [CK99] in a complete market, and by Cvitanic [Cvi00] in incomplete and constrained markets. In both cases, they use the duality approach, as opposed to the Neyman-Pearson lemma approach, employed by Föllmer and Leukert [FL00] to solve the same problem in an unconstrained incomplete market.

Choosing $\nu(x) = (x^-)^p$, with $p > 1$, one obtains a solution for the problem studied by Pham [Pha00] in discrete time.

Example 3.19 *(Utility maximization).* Let U be a concave bounded increasing function. The *utility maximization* problem

$$\max_{\theta \in \Theta_C^1} E\left[U\left(V_T^c(\theta) - H(\theta)\right)\right]$$

admits a solution. In fact, apply Proposition 3.17, with $\nu(x) = -U(x)$. This problem has been studied for European options in a Markovian model by Hodges and Neuberger [HN89] and developed more rigorously by Davis, Panas, and Zariphopoulou [DPZ93]: in both papers, a stochastic control problem is considered, and the assumptions on the model lead to a Hamilton-Jacobi-Bellmann equation which can be solved in a weak sense. In more general models, the same problem has been studied in the frictionless case by Cvitanic and Karatzas [CK96] and by Kramkov and Schachermayer [KS99] with the convex duality approach.

Remark 3.20. The variable H needs not be a function of X_T alone: in fact we only require that it is $\mathcal{F}$-measurable. This means that the existence result is valid for a general path-dependent option, as long as its exercise is fixed at time T. This excludes American-type options.

3.3 Constrained Problems

In this section we study the problem of hedging with constraints on the space of strategies. Essentially, we consider two types of constraints: those on the position in the risky asset, such as limits on short-selling, and those on the portfolio value, such as budget constraints (which here play the role of constraints, rather than assumptions on admissible strategies).

The existence of a constrained minimum depends on two conditions: the stability of the restricted set of admissible strategies under the transformations used on minimizing sequences, and its closedness in the topology where the risk functional is lower semicontinuous.

In this setting, it becomes evident that the more transformations are used in the proof of the unconstrained problem, the smaller is the set of tractable constraints. Since in the case of L^p we only take convex combinations of strategies, it follows that we can obtain an existence result for constraints of the type $\theta_t \in K$, where $K_t(\omega)$ is a closed convex subset of $\mathbb{R}^d$.

On the contrary, in the L^1 case we also use stopped strategies, hence $K_t(\omega)$ will have to be a closed convex containing zero. At any rate, it seems that these conditions are not restrictive for most applications.

To formalize these requirements, we give the following:

Definition 3.21. *Given $K \subset \mathbb{R} \times \Omega \times \mathbb{R}^d$, we we denote by $K_t(\omega)$ the section of K with respect to the first two components (t, ω).*

We say that $\mathcal{K}$ is a convex constraint *if:*

i) K is measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}$ is the predictable σ-algebra on $\mathbb{R} \times \Omega$, and $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra;

ii) $K_t(\omega)$ is convex a.s. in $dtdP$;

iii) $K_t(\omega)$ is closed a.s. in $dtdP$.

We say that a strategy θ satisfies the constraint $\mathcal{K}$ *if* $\theta_t(\omega) \in K_t(\omega)$ *a.s. in* $dtdP$.

It is easily checked that:

Corollary 3.22. *If* K *is a convex constraint, we have that:*

i) If θ *and* η *satisfy* K*, then for each* $\lambda \in (0,1)$ *the strategy* $(1-\lambda)\theta + \lambda\eta$ *satisfies* K*;*

ii) If $\{\theta^n\}_{n\in\mathbb{N}}$ *satisfy* K *and* $\theta^n \to \theta$ *a.s. in* $dtdP$*, then* θ *satisfies* K.

The next proposition establishes the existence of a minimum for convex constraints.

Proposition 3.23. *Let* K *be a convex constraint. Denoting by*

$$\Gamma^p(K) = \{\theta_t \in \Theta^p_C : \theta \textit{ satisfies } K\}$$

if $\Gamma^p(K)$ *is not empty, we have that:*

i) if $p > 1$*, then for all* $M > 0$ *the minimum problem*

$$\min_{\theta \in \Theta^p_{C,M} \cap \Gamma^p(K)} \rho\left(V^c_T(\theta) - H(\theta)\right)$$

admits a solution.

ii) if $p = 1$*, and* $0 \in K_t(\omega)$ $dtdP$*-a.s., then for all* $M > 0$ *the minimum problem*

$$\min_{\theta \in \Theta^1_{C,M} \cap \Gamma^1(K)} \rho\left(V^c_T(\theta) - H(\theta)\right)$$

admits a solution.

Proof. Reread the proof of Propositions 3.15 and 3.16, observing that convex combinations of strategies in $\Gamma^p(K)$ remain in $\Gamma^p(K)$ and that (for 3.16) a strategy in $\Gamma^1(K)$, if stopped, remains in $\Gamma^1(K)$. Finally, if $\theta^n \to \theta$ a.s. in $dtdP$, and $\theta^n \in \Gamma^p(K)$, then $\theta \in \Gamma^p(K)$. $\square$

Example 3.24 *(Short-selling).* For $K = \{(t,\omega,x) : x_i \geq 0 \ \forall\ i \in \{1 \ldots d\}\}$, the constraint above becomes $\theta_t \geq 0$ for all t, which amounts to forbid the short sale of X. Notice that in this case the constraint is deterministic.

Example 3.25 *(Budget Constraints).* Consider a solvability condition, which in our notation looks like:

$$V_t^c(\theta) - k_t X_t |\theta_t| \geq -l_t$$

where l_t is the maximum credit line available at time t. This constraint is clearly stable with respect to the operations considered, but apparently it does not fit in Proposition 3.23, where K should be written explicitly. Indeed, in this example it can be shown that:

$$K_t(\omega) = \begin{cases} \mathbb{R}^d & \text{if } t < \tau \\ 0 & \text{if } t \geq \tau \end{cases}$$

where τ is defined as:

$$\tau = \inf\{t : \ V_t^c(\theta) - k_t X_t |\theta_t| = -l_t\}$$

In other words, the agent is unconstrained until the solvency limit is hit, then the position must be closed for the rest of the period.

As mentioned in Remark 1.15, the presence of budget constraints allows to consider a larger class of risk functionals. In fact we have the following:

Theorem 3.26. *Let ρ be a σ-additive coherent risk measure (see Delbaen [Del00] for details). In other words, $\rho(X) = \lim_{k\to\infty} \sup_{P\in\mathcal{P}} E_P[-(X \wedge k)]$, where $\mathcal{P}$ is a set of probabilities, all absolutely continuous with respect to P.*

If K is defined as in Example 3.25, H is bounded, and $\mathcal{P}$ is weakly relatively compact, the same minimum problems as in Proposition 3.23 admit a solution.

We recall the following Lemma from Delbaen [Del00]:

Lemma 3.27. *Let $\mathcal{P}$ be a weakly relatively compact set of absolutely continuous probabilities. If X_n is uniformly bounded in $L^\infty(\Omega)$ and $X_n \to X$ a.s., then* $\lim_{n\to\infty} \rho(X_n) = \rho(X)$.

Proof. of Theorem 3.26 A coherent risk measure has all the properties of a convex risk functional, except that the Fatou property is satisfied only by bounded sequences of random variables. As a result, the semicontinuity

Lemma 3.12 generally fails. We now show that, given a minimizing sequence θ^n, under the additional assumptions above we can obtain another minimizing sequence η^n such that $F : \theta \mapsto \rho(V_T^c(\theta) - H(\theta))$ is lower semicontinuous with respect to η^n.

Let θ^n be a minimizing sequence, and denote by $\overline{\rho} = \inf_n F(\theta^n)$. As in the proofs of Propositions 3.15 and 3.16, up to a subsequence of convex combinations we can assume that $V_T^c(\theta^n) \to V_T^c(\theta)$ a.s. for some admissible strategy θ. Also, by definition of ρ, for each ε there exists some n and k_n such that $\rho((V_T^c(\theta^n) - H(\theta^n)) \wedge k) < \overline{\rho} + \varepsilon$ for all $k > k_n$.

By the assumptions on K, $V_T^c(\theta^n) > -l$ for all n. Define now the stopping times $\tau_k = \inf\{t : \ V_t^c(\theta) \geq k\}$. By construction, for the stopped strategies $\eta_t^{n,k} = \theta_{t\wedge\tau_k}^n$ and $\eta_t^k = \theta_{t\wedge\tau_k}$, we have that $V_T^c(\eta^{n,k}) \leq k$. Now, the sequence $V_T^c(\eta^{n,k})$ is uniformly bounded in L^∞, $\mathcal{P}$ is weakly relatively compact, and $\lim_{n\to\infty} V_T^c(\eta^{n,k})) = V_T^c(\eta^k)) = V_T^c(\theta) \wedge k$ a.s. By the above lemma we have that

$$\lim_{n\to\infty} \rho(V_T^c(\eta^{n,k}) - H(\eta^{n,k})) = \rho(V_T^c(\eta^k) - H(\eta^k)) = \rho(V_T^c(\theta) \wedge k - H(\eta^k))$$

and, since H is bounded,

$$\begin{aligned} \lim_{k\to\infty} \rho(V_T^c(\eta^k) - H(\eta^k)) &= \lim_{k\to\infty} \rho(V_T^c(\theta) \wedge k - H(\eta^k)) = \\ &= \lim_{k\to\infty} \rho((V_T^c(\theta) - H(\eta^k)) \wedge k) = \rho(V_T^c(\theta) - H(\eta)) \end{aligned}$$

Therefore η^k is a minimizing sequence, and F is continuous with respect to it. $\square$

3.4 The Semimartingale Case

In this section we discuss the problems arising in the more realistic case where X is a semimartingale, and give an extension of the foregoing results to this setting.

The main idea is to reduce to the previous martingale case, by the well-known technique of a change of measure. To exclude arbitrage, throughout this section we make the following:

Assumption 3.28. *X is a continuous semimartingale which admits at least a local martingale measure.*

As anticipated in chapter 1, the main difference from the martingale case is that the space:

$$\Theta^p = \{\theta : \theta\ \mathcal{F}_t\text{-predictable}, (\theta \cdot X)_T \in L^p(P)\}$$

in general may not be closed, and hence is not suitable for studying optimization problems. On the contrary, the space of contingent claims K_p proposed by Delbaen and Schachermayer [DS96] provides closure properties analogous to the martingale case. This means that the natural extension of Θ^p_C to the semimartingale case is given by:

$$\Theta^p_C(P) = \{\theta : G_T(\theta) \in K_p, C_T(\theta) \in L^p(P)\}$$

We now see how a minimization problem of the type:

$$\min_{\theta \in \Theta^p_C(P)} \rho(V^c_T(\theta) - H(\theta))$$

fits into the framework outlined in the previous sections. We have the following:

Proposition 3.29. *Let $1 \leq p \leq \infty$, X a continuous semimartingale locally in $L^p(P)$, and $Q \in \mathcal{M}^e_{p'}(P)$. Then the problem:*

$$\min_{\theta \in \Theta^p_{C,M}(P)} \rho(V^c_T(\theta) - H(\theta)) \tag{S}$$

admits a solution for any $M > 0$.

Proof. Since the set $\{\frac{dQ}{dP} : Q \in \mathcal{M}^e_{p'}(P)\}$ is convex and closed in $L^{p'}(P)$, it follows that there exists a countable set of martingale measures $\{Q_i\}_i$, such that $\left\{\frac{dQ_i}{dP}\right\}_i$ is dense in $\{\frac{dQ}{dP} : Q \in \mathcal{M}^e_{p'}(P)\}$ in the $L^{p'}(P)$ norm.

By the Hölder inequality, the identity map $Id : K_p(P) \mapsto G_T(\Theta^1(Q_i))$ is a continuous operator for all i. Note also that if $C_T(\theta) \in L^p(P)$, then $C_T(\theta) \in L^1(Q_i)$.

Let θ^n be a minimizing sequence for (S). Since the set of simple strategies is dense both in $K_p(P)$ and $\Theta^1_{C,M'}(Q_i)$, it follows that θ^n is a minimizing sequence for the problem:

$$\min_{\theta \in \Theta^1_{C,M'}(Q_i)} \rho(V^c_T(\theta) - H(\theta)) \tag{M}$$

for a suitable M'. By Proposition 3.16 we can extract a minimizing sequence of convex combinations of θ^n converging to a minimizer θ of (M). Since minimizing sequences are stable under convex combinations, we can take further subsequences of convex combinations converging to θ in $L^1(Q_i)$ for any finite set of i. By a diagonalization argument, we obtain a η^n such that $\eta^n \to \theta$ in $L^1(Q_i)$ for all i.

If $Q \in \mathcal{M}^e_{p'}(P)$, we can assume up to a subsequence that $\frac{dQ_i}{dP} \to \frac{dQ}{dP}$ in $L^{p'}(P)$. By construction, for all i we have that:

$$E_{Q_i}\left[G_T(\theta)\right] = 0$$

and as $i \to \infty$, we obtain that $E_Q\left[G_T(\theta)\right] = 0$ for any $Q \in \mathcal{M}^e_{p'}(P)$. By Theorem 1.8, it follows that $\theta \in K_p$. Also, Lemma 3.8 implies that $C_T(\theta) \in L^p(P)$. Therefore $\theta \in \Theta^p_C(P)$, and the proof is complete. □

Remark 3.30. The diagonalization procedure in Proposition 3.29 is necessary since the map $T : K_p(P) \mapsto G_T(\Theta^1(Q))$ is generally not onto (see Delbaen and Schachermayer [DS96], Remark 2.2 c) for details).

Chapter 4

Mean-Variance Analysis and Transaction Costs

In the previous chapters we considered a large class of risk functionals, imposing some natural economic requirements such as nonsatiation and risk-aversion. However, some classical problems in finance do not fit in this framework. The most prominent example is probably *mean-variance analysis*.

In problems of mean-variance optimization, economic agents seek to minimize the quadratic norm of their final position, possibly under constraints on expected return. This approach has obvious advantages in terms of tractability, although the choice of variance as a risk measure lends itself to the obvious critique that gains as well as losses are equally penalized. In other words, the nonsatiation requirement is not satisfied.

Nevertheless, this is generally a harmless argument, since mean-variance analysis often offers insights that are valid in more general settings. Indeed, all classical Finance Theory was first built upon the mean-variance criterion, and several results were later extended to the context of utility functions.

4.1 The Dissipation effect

Our goal is to show that the analogy between mean-variance analysis and the ordinary utility maximization breaks down in presence of transaction costs.

Intuitively, transaction costs alter the structure of the problem, allowing agents to freely dispose of their excess capital. As a result, choosing a risk measure which does not fulfill the nonsatiation requirement will lead to "optimal" strategies which generate fictitious trading, which has no other purpose than avoiding penalized gains.

This phenomenon occurs in a subtle manner. Indeed, our parametrization of trading strategies is based on number of shares, rather than on cumulative purchases and sales, in an effort to avoid suboptimal strategies (which, by the way, become desirable in the mean-variance setting). Thus, rather than devising directly a riskless dissipating strategy, it is easier to consider a sequence of strategies which delivers the same result in the limit.

Mathematically, we can observe the following: the quadratic norm fails to satisfy the lower semicontinuity property with respect to the a.s. convergence in $dtdP$. As a result, some minimizing sequences do not converge to minimizers, but to "suboptimal" strategies, which do not meet the optimality requirement because of their lack of dissipation.

Since this chapter is mainly devoted to exhibit negative results, we shall not pursue full generality. In fact, examples will be limited to square-integrable continuous semimartingales.

The next proposition provides the existence of minimizing strategies used in the subsequent examples:

Proposition 4.1. *Let X_t be a continuous process. For each increasing predictable process C_t there exists a sequence of strategies θ^n such that:*

$$\lim_{n\to\infty}\int_{[0,T]}\theta^n_t dX_t = 0 \quad a.s. \qquad and \qquad \lim_{n\to\infty}\int_{[0,T]} X_t d|D\theta^n_t| = C_T \quad a.s.$$

and hence:

$$V^c_T(\theta^n) \to c - kC_T \quad a.s.$$

In addition, if C_T and X_T are p-integrable, then:

$$E\left[|V^c_T(\theta^n)|^p\right] \to E\left[|c - kC_T|^p\right]$$

Proof. Define the approximate process

$$C^n_t = C_{\lfloor t2^n\rfloor 2^{-n}}$$

It is clear that C^n converges to C a.s. in $dPdt$ as $n \to \infty$. Denoting by δ_x the usual Dirac measure on the point x, we define θ^n in terms of its derivative:

$$D\theta^n = \sum_{k=1}^{2^n T} \frac{1}{2} \left(C_{k2^{-n}} - C_{(k-1)2^{-n}} \right) \left(\frac{\delta_{k2^{-n}}}{X_{k2^{-n}}} - \frac{\delta_{(k+\frac{1}{2})2^{-n}}}{X_{(k+\frac{1}{2})2^{-n}}} \right)$$

and hence $\theta_t = D\theta([0,t])$. since X is continuous, it is easy to see that, for a.e. ω, $D\theta^n$ converges to the null measure in the weak star topology. Hence we have:

$$\lim_{n\to\infty} \int_{[0,T]} \theta_t^n dX_t = \lim_{n\to\infty} \int_{[0,T]} (X_T - X_t) dD\theta_t^n = 0 \qquad \text{a.s.}$$

which proves the first part of the statement. Note also that:

$$|D\theta^n| = \sum_{k=1}^{2^n T} \frac{1}{2} \left(C_{k2^{-n}} - C_{(k-1)2^{-n}} \right) \left(\frac{\delta_{k2^{-n}}}{X_{k2^{-n}}} + \frac{\delta_{(k+\frac{1}{2})2^{-n}}}{X_{(k+\frac{1}{2})2^{-n}}} \right)$$

and therefore:

$$\int_{[0,T]} X_t d|D\theta_t^n| = \sum_{k=1}^{\lfloor 2^n T \rfloor} \left(C_{k2^{-n}} - C_{(k-1)2^{-n}} \right) = C_{\lfloor T2^n \rfloor 2^{-n}}$$

which proves that convergence holds a.s. The last part of the thesis follows by dominated convergence. □

As a corollary of the above proposition, we obtain that the "optimal" minimum-variance portfolio under transaction costs consists in dissipating all initial wealth. This shows that the mean-variance criterion leads to paradoxical results, when combined with transaction costs.

Corollary 4.2. *Let X be a continuous, square-integrable semimartingale. Then we have:*

$$\inf_{\theta \in \Theta_C^2} E\left[(V_T^c(\theta))^2 \right] = 0$$

Proof. The infimum is obviously nonnegative. By Proposition 4.1, there exists a sequence θ^n such that $\int_{[0,T]} \theta_t^n dX_t \to 0$ and $\int_{[0,T]} X_t d|D\theta_t^n| \to -\frac{c}{k}$ a.s. (and hence $V_T^c(\theta^n) \to 0$ a.s.). Since X is square-integrable, convergence holds in L^2, θ^n is a sequence of minimizing strategies, and the infimum is 0. □

Remark 4.3. The possibility to dissipate wealth is not an exclusive feature of markets with transaction costs. In fact, it has been long recognized that in frictionless markets some strategies may lead to sure losses, and this can be achieved through reverse doubling strategies (see for instance Schachermayer [Sch92] for an explicit example, and Delbaen and Schachermayer [DS97] for a thorough treatment of this subject).

However, these dissipating strategies are fundamentally different from those presented here, and indeed forbidden in the context of frictionless mean-variance hedging. In fact, without transaction costs the space Θ^2_C reduces to a set of strategies leading to stochastic integrals bounded in L^2. This condition in turn implies the uniform integrability of $V^c_T(\theta)$. As a result, $E_Q\left[V^c_T(\theta)\right] = c$ for some martingale measure Q, and this in not compatible with the possibility that $V^c_T(\theta) < c$ a.s.

Corollary 4.2 is the easiest example of the dissipation effect, but this phenomenon occurs in full generality with *mean-variance hedging.* We now provide a precise statement, which can also be seen as a further argument in favor of decreasing risk functionals:

Proposition 4.4. *Let X be a continuous, square-integrable semimartingale, and $H \in L^2(\mathcal{F}_T, P)$ a random variable. Then we have:*

$$\inf_{\theta\in\Theta^2_C} E\left[(V^c_T(\theta) - H)^2\right] = \inf_{\theta\in\Theta^2_C} E\left[\left[(V^c_T(\theta) - H)^-\right]^2\right] \tag{4.1}$$

and hence, if there exists an optimal strategy θ for the problem

$$\min_{\theta\in\Theta^2_C} E\left[(V^c_T(\theta) - H)^2\right] \tag{4.2}$$

then

$$P(V^c_T(\theta) > H) = 0$$

Proof.
Denote $l = \inf_{\theta\in\Theta^2_C} E\left[(V^c_T(\theta) - H)^2\right]$ and $m = \inf_{\theta\in\Theta^2_C} E\left[\left[(V^c_T(\theta) - H)^-\right]^2\right]$. We obviously have that $l \geq m$, so we need to show that the reverse inequality holds.

Let η be a strategy such that $E\left[(V_T^c(\eta)-H)^2\right]<l+\varepsilon$. We claim that $V_t^c(\eta)$ converges to $V_T^c(\eta)$ in $L^2(P)$ as $t\to T$. In fact, we have that:

$$\begin{aligned}|V_T^c(\eta)-V_t^c(\eta)| = &\left|\int_{(t,T]}\eta_s dX_s - k\int_{(t,T]} X_s d|D\eta|_s - k\left(|\eta_T|X_T-|\eta_t|X_t\right)\right| \leq \\ &\leq \left|\int_{(t,T]}\eta_s dX_s\right| + k\left|\int_{(t,T]} X_s d|D\eta|_s\right| + \\ &\qquad + k\,|\eta_T|\,|X_T-X_t| + kX_t\,||\eta_T|-|\eta_t|| \end{aligned} \tag{4.3}$$

Denoting the Doob-Meyer decomposition $X=M+A$, for the first term we have:

$$\begin{aligned} E\left[\left|\int_{(t,T]}\eta_s dX_s\right|^2\right] &= E\left[\left|\int_{(t,T]}\eta_s dM_s + \int_{(t,T]}\eta_s dA_s\right|^2\right] \leq \\ &\leq 2\left(E\left[\left|\int_{(t,T]}\eta_s dM_s\right|^2\right] + E\left[\left|\int_{(t,T]}\eta_s dA_s\right|^2\right]\right) = \\ &= 2\left(E\left[\int_{(t,T]}\eta_s^2 d\langle M\rangle_s\right] + E\left[\left|\int_{(t,T]}\eta_s dA_s\right|^2\right]\right)\end{aligned}$$

By the continuity of X, $d\langle M\rangle_s$ is diffuse, and by the existence of a martingale measure, $dA_t \ll d\langle M\rangle_t$, hence convergence follows. Analogously, the third term in (4.3) converges. For the second and the last terms, convergence follows from the left continuity of η at T, which holds by assumption.

The above argument shows that for some $\delta>0$ we have:

$$\begin{aligned} E\left[\left(V_{T-\delta}^c(\eta)-H\right)^2\right] &< l+2\varepsilon \quad \text{and} \\ E\left[\left(V_T^c(\eta)-V_{T-\delta}^c(\eta)\right)^2\right] &< \varepsilon \end{aligned}$$

Analogously, we can assume that:

$$E\left[(H-E\left[H|\,\mathcal{F}_{T-\delta}\right])^2\right]<\varepsilon$$

Now we define the strategy ζ as follows: up to time $T-\delta$ we set $\zeta_t=\eta_t$. From time $T-\delta$ onwards, we follow a strategy provided by Proposition 4.1, which delivers a final payoff $V_{T-\delta}^c+\chi$ such that

$$E\left[\chi|\,\mathcal{F}_{T-\delta}\right] = -\max(0, V_{T-\delta}^c(\eta)-E\left[H|\,\mathcal{F}_{T-\delta}\right])$$

and the variance of χ is smaller than ε.

Thus, denoting by $A = \{V^c_{T-\delta}(\eta) - E\left[H|\, \mathcal{F}_{T-\delta}\right] > 0\}$ we obtain:

$$
\begin{aligned}
E\left[\left(V^c_T(\zeta) - H\right)^2\right] &= E\left[\left(V^c_{T-\delta}(\eta) + \chi - H\right)^2\right] = \\
&= E\left[E\left[\left(H - E\left[H|\,\mathcal{F}_{T-\delta}\right]\right)^2 1_A \middle|\, \mathcal{F}_{T-\delta}\right]\right] + E\left[\left(V^c_{T-\delta}(\eta) - H\right)^2 1_{\Omega\setminus A}\right] \leq \\
&\leq \varepsilon + E\left[\left(V^c_{T-\delta}(\eta) - H\right)^2 1_{\Omega\setminus A}\right] < 2\varepsilon + E\left[\left(V^c_T(\eta) - H\right)^2 1_{\Omega\setminus A}\right]
\end{aligned}
$$

Also, note that as $\delta \to 0$ the indicator of the set A converges a.s. to the indicator of the set $\{V^c_T(\eta) > H\}$, and hence for a small enough choice of δ we obtain:

$$
\begin{aligned}
E\left[\left(V^c_T(\zeta) - H\right)^2\right] \leq &3\varepsilon + E\left[\left(V^c_T(\eta) - H\right)^2 1_{\{V^c_T(\eta) \leq H\}}\right] = \\
= &3\varepsilon + E\left[\left[\left(V^c_T(\eta) - H\right)^-\right]^2\right]
\end{aligned}
$$

The above discussion shows that from any minimizing sequence η^n, such that $\inf_n E\left[\left(V^c_T(\eta^n) - H\right)^2\right] = l$ we can construct another sequence ζ^n such that $\inf_n E\left[\left[\left(V^c_T(\eta^n) - H\right)^-\right]^2\right] = l$. This proves that $l \leq m$.

Suppose that θ is a minimizer for problem 4.2. Then we must have that $P(V^c_T(\theta) > H) = 0$, otherwise $l > m$, which is absurd. □

4.2 Notes

Remark 4.5. The difficulty to apply the mean-variance criterion to the transaction costs setting was already recognized in discrete time by Motoczinsky [Mot99]. He considered two different mean-variance hedging problems: one on a general space of strategies Θ, allowing dissipation, and one on the space Θ_0 of "reasonable" strategies, where simultaneous buying and selling is forbidden.

He proved the existence of a solution in the space Θ, and observed that for some contingent claims the infimum on Θ_0 may not coincide with the minimum of Θ.

Our results suggest that the gap between the two infima should not be large, and primarily due to the discrete-time setting, which somehow reduces the ability for achieving dissipation by means of "reasonable" strategies. In other words, if an agent is forbidden to buy and sell at the same time in discrete time, he will try to circumvent such restriction by buying more, only to sell the excess amount at a nearby date. This procedure will create some extra noise, but as the time step becomes smaller and smaller it will enable the agent to effectively avoid the ban on dissipation.

Chapter 5

Problems with Budget Constraints

While the approach of imposing integrability conditions to avoid arbitrage is very natural in the context of quadratic risk criteria, the arguments given in the previous chapter show that this criterion has some drawbacks, especially in combination with transaction costs.

Also, there are compelling economic arguments in favor of arbitrage restrictions which do not depend on the probability measure used by a particular agent, but only on its equivalence class. For instance, an investor may summarize his views with a probability measure P, and devise some strategy θ which optimizes his goal. However, in order to implement this strategy he may need to borrow from a bank, which has a different measure $\tilde{P}$. While the investor and his bank will generally disagree on the probability of a given event, we can expect that they agree on which events should be deemed as impossible, and define admissible strategies only in terms of statements which can be agreed upon.

This approach leads to definition 1.9 of admissible strategies, which has been widely adopted in Mathematical Finance. In this chapter we study problems of optimal investment on this class of strategies, removing many of the restrictions made in chapter 3. In particular, we relax the continuity assumption on the process X, allowing for quasi-left continuous processes. More importantly, we also drop the semimartingale assumption.

The quasi-left continuity condition essentially means that the jumps of the asset process X occur at totally inaccessible stopping times. In other words, any attempt of predicting a jump with the information available before is deemed to fail almost surely. Some financial risks do not typically satisfy this requirement: for instance, interest rate changes are generally announced at previously scheduled meetings, which are not only accessible but indeed predictable. However, this difficulty can be generally overcome with small model modifications, without altering the structure of the problems: in the example of interest rates, one could replace the predictable jump with a totally inaccessible jump uniformly distributed in an arbitrarily small time interval, thus preserving quasi-left continuity.

Dropping the semimartingale assumption requires particular care for its arbitrage consequences, and in chapter 2 we have seen some criteria to stay on the safe side. However, it turns out that ensuring no-arbitrage is not enough in presence of transaction costs, as this does not guarantee the compactness of the space of admissible strategies. To achieve this, we need the following (see also Proposition 2.16 for the relationship with the discrete time condition introduced by Schachermayer [Sch01a]):

Definition 5.1. *We say that a market with transaction costs k is* compact arbitrage free *if it γ-arbitrage free and:*

$$\min_{1\leq i\leq d} \operatorname*{essinf}_{t\in[0,T]}(k_t^i - \gamma_t^i) > 0 \quad a.s.$$

Essentially, the above condition requires a minimal size for transaction cost, so that if a strategy is to be admissible, then it must lead to a bounded (in a sense to be made precise) trading volume. This is basically the key idea to the following section.

5.1 Compactness of admissible strategies

Our general approach to prove the existence of optimal strategies remains the direct method, therefore we still look for compactness and semicontinuity results.

The main difference with chapter 3 is that the weak convergence in L^p will now be replaced with convergence in probability, which does not depend on the particular measure P.

We start our discussion with the following:

Definition 5.2. *We say that a sequence of processes θ^n converges to θ* countably almost surely *(c.a.s.) if for almost every ω there exists a countable set $N(\omega) \subset [0,T]$ such that $\theta^n(\omega) \to \theta(\omega)$ for all $t \notin N(\omega)$.*

Remark 5.3. It is immediately seen that convergence c.a.s. implies a.s. convergence in $dtdP$, and indeed we will only need the latter in the main results of this chapter. However, the more precise result of c.a.s. convergence will allow an alternative and simpler proof for the semimartingale case, and this is the reason why we introduce the above definition.

The main result of this section is the following:

Proposition 5.4. *Let X be compact arbitrage free and satisfy 1.10. If $\theta^n \in \mathcal{A}_c^k$ is a sequence of finite variation processes, then there exists a sequence $\eta^n \in conv(\theta_n, \theta_{n+1}, \dots)$ such that η^n converges c.a.s. to a finite variation process $\theta \in \mathcal{A}_c^k$.*

We break the proof of Proposition 5.4 into three parts. First we recall the following Lemma from Delbaen and Schachermayer [DS94]:

Lemma 5.5 ([DS94], Lemma A1.1). *Let $(f_n)_{n\geq 1}$ be a sequence of $[0,\infty)$ valued measurable functions on a probability space $(\Omega, \mathcal{F}, P)$. There exists a sequence $g_n \in conv(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n\geq 1}$ converges almost surely to a $[0,\infty]$ valued function g.*

If $conv(f_n, f_{n+1}, \dots)$ is bounded in L^0, then g is finite almost surely. If there are $\alpha > 0$ and $\delta > 0$ such that for all n: $P(f_n > \alpha) > \delta$, then $P(g > 0) > 0$.

The following Lemma can be seen as a compactness result for Fatou convergence (see Kramkov [Kra96], Lemma 4.2) but here convergence is sought within the class of predictable processes (see also Kabanov and Last [KL02], Lemma 3.4 for a similar result in the context of transaction costs):

Lemma 5.6. *Let θ^n be a sequence of processes of finite variation, such that the set conv$(\{\|D\theta^i\|_T\}_{i\in\mathbb{N}})$ is bounded in $L^0(\Omega)$.*

Then there is a sequence $\eta^n \in$ conv$(\theta_n, \theta_{n+1}, \dots)$ such that η^n converges c.a.s. to a finite variation process θ.

Proof. By the Hahn decomposition, a function of bounded variation is a difference of two monotone functions. Hence we can write $\theta^n_t = L^n_t - M^n_t$, where L^n and M^n are increasing processes which are essentially unique under the condition that $|D\theta^n|_t = L^n_t + M^n_t$.

For a given time t, we have that:

$$\sum_{k\geq n} \alpha_k L^k_t \leq \sum_{k\geq n} \alpha_k |D\theta^k|_t \leq \sum_{k\geq n} \alpha_k |D\theta^k|_T \qquad \text{a.s.}$$

therefore conv$(L^n_t, L^{n+1}_t, \dots)$ is bounded in L^0. By Lemma 5.5, up to a sequence of convex combinations we can assume that L^n_t converges almost surely to some real valued variable L_t.

By a diagonalization argument, up to a sequence of convex combinations we can assume that, for all $t \in \{0,T\} \cup (\mathbb{Q} \cap (0,T))$, L^n_t converges almost surely to a process $(L_t)_{t\in\mathbb{Q}\cap[0,T]}$. Clearly, L_t is an increasing process.

We define $\tilde{L}_t = \sup_{s\in\mathbb{Q}\cap(0,t)} L_s$. Since $\tilde{L}$ is left-continuous, it is obviously predictable. We now show that, for each ω, $L^n_t \to \tilde{L}_t$ everywhere but in the discontinuity points of $\tilde{L}(\omega)$, which are at most a countable set. In fact, if $L(\omega)$ is continuous in t, for any $\varepsilon > 0$ we can find $p, q \in \mathbb{Q} \cap [0,T]$ such that $p < t < q$ and $L_p(\omega) \leq L_q(\omega) \leq L_p(\omega) + \varepsilon$. Passing to the limit, we get:

$$L_p(\omega) \leq \liminf_{n\to\infty} L^n_t(\omega) \leq \limsup_{n\to\infty} L^n_t(\omega) \leq L_p(\omega) + \varepsilon$$

Since ε is arbitrary, it follows that L^n_t converges to $\tilde{L}_t$.

Repeating the same argument for M, we obtain that, up to a sequence of convex combinations, L^n and M^n respectively converge c.a.s. to increasing processes $\tilde{L}$ and $\tilde{M}$. As a result, $L^n_t - M^n_t$ converges to $\theta_t = \tilde{L}_t - \tilde{M}_t$, which is a finite variation process. $\square$

So far no reference is present to the asset process. The assumptions on X in Proposition 5.4 will now be needed to link the admissibility of trading strategies to the boundedness condition in Lemma 5.6.

The following lemma must be well-known, but since we have no reference we report its short proof.

Lemma 5.7. *Let X and $\{Y_i\}_{i\in I}$ be strictly positive, finite valued, random variables. Then $\{Y_i\}_{i\in I}$ is bounded in L^0 if and only if $\{XY_i\}_{i\in I}$ is bounded in L^0.*

Proof. Suppose that $\{Y_i\}_{i\in I}$ is bounded in L^0. We obviously have:

$$P(XY_i > M) = P(XY_i > M, X \leq N) + P(XY_i > M, X > N) \leq$$
$$\leq P(Y_i > \frac{M}{N}) + P(X > N)$$

With a suitable choice of M and N, both these terms are arbitrarily small, as X is finite valued, and $\{Y_i\}_{i\in I}$ is bounded in L^0.

The reverse implication follows from the first one, denoting $X' = \frac{1}{X}$ and $Y'_i = XY_i$. □

Remark 5.8. An immediate consequence of the above lemma is that boundedness in L^0 is invariant under a change to an equivalent measure.

Proof of Proposition 5.4. For any $\theta \in \mathcal{A}^k_c$, we have:

$$\begin{aligned} -c \leq (\theta\cdot X)_T - \int_{[0,T]} k_s X_s \cdot d|D\theta|_s - kX_T\cdot|\theta_T| = \\ = (\theta\cdot X)_T - \left(\int_{[0,T]} \gamma_s X_s \cdot d|D\theta|_s + \gamma_T X_T \cdot |\theta_T|\right) - \\ - \left(\int_{[0,T]} (k_s-\gamma_s) X_s \cdot d|D\theta|_s + (k_T-\gamma_T) X_T \cdot |\theta_T|\right) \end{aligned}$$

Integrating the second term by parts, and recalling the that X is γ-arbitrage free, we obtain that (see also Lemma 2.9):

$$(\theta\cdot X)_T - \left(\int_{[0,T]} \gamma_s X_s \cdot d|D\theta|_s + \gamma_T X_T \cdot |\theta_T|\right) \leq (\theta\cdot\tilde{X})_T$$

and hence:

$$-c \leq (\theta\cdot\tilde{X})_T - \left(\left(\min_{1\leq i\leq d} \operatorname*{essinf}_{t\in[0,T]}(k^i_t-\gamma^i_t)\right)\xi_T\cdot|D\theta|_T + (k-\gamma)X_T\cdot|\theta_T|\right)$$

where $\xi_t = \inf_{s\in[0,t]} X_s$. Under the measure Q, the stochastic integral $(\theta\cdot\tilde{X})_T$ is a local martingale bounded from below, thus a supermartingale. Taking expectations, we have:

$$E\left[\xi_T \cdot |D\theta|_T + X_T \cdot |\theta_T|\right] \leq \frac{c}{\min_{1\leq i\leq d} \operatorname{essinf}_{t\in[0,T]}(k_t^i - \gamma_t^i)}$$

This inequality is clearly stable under convex combinations of strategies, and therefore the closed convex hull of the set $\{\xi_T \cdot |D\theta|_T : \theta \in \mathcal{A}_c^k\}$ is bounded in $L^1(Q)$ and hence in $L^0(Q)$.

Note that $\xi_T > 0$ a.s. To see this, denote by $\tau = \inf\{t : X_t = 0 \text{ or } X_{t^-} = 0\}$. Since X is γ-arbitrage free, we have that $\tau = \inf\{t : \tilde{X}_t = 0 \text{ or } \tilde{X}_{t^-} = 0\}$. But then we obtain that $\tau > T$ a.s., as $\tilde{X}$ is a strictly positive martingale under Q.

This implies that we can apply Lemma 5.7, and we obtain that the closed convex hull of $\{\|D\theta\|_T : \theta \in \mathcal{A}_c^k\}$ is bounded in $L^0(P)$.

Now, let $\theta^n \in \mathcal{A}_c^k$ be a sequence of finite variation processes. From the above discussion it follows that the assumptions of Lemma 5.6 are satisfied, therefore we can assume, up to a sequence of convex combinations, that θ^n converges c.a.s. to some finite variation process θ. The admissibility of θ follows from Proposition 5.13 (or Proposition 5.10, if X is continuous). $\square$

5.2 Existence of optimal strategies

Now we need the lower semicontinuity of risk functionals with respect to c.a.s. convergence. For convex decreasing functionals, it will be enough to check that the portfolio value V_t^c is upper semicontinuous, and this is the point where we shall need the quasi-left continuity of X.

We begin with a lemma which links the pointwise convergence of $\theta_t^n(\omega)$ to the weak star convergence of the measures $D\theta^n(\omega)$.

Lemma 5.9. *If conv($\{\|D\theta^i\|_T\}_{i\in\mathbb{N}}$) is bounded in L^0 and $\theta_t^n \to \theta_t$ a.s. in $dtdP$, then up to a sequence of convex combinations $D\theta_t^n \rightharpoonup D\theta_t$ for a.e. ω.*

Proof. By Lemma 5.5, up to a sequence of convex combinations we can assume that $\lim_{n\to\infty} \|D\theta^n\|_T = V(\omega)$, with $V < \infty$ a.s.

We have:

$$\{\limsup_{n\to\infty} \|D\theta^n\|_T > M\} = \{\liminf_{n\to\infty} \|D\theta^n\|_T > M\} = \bigcup_k \bigcap_{n\geq k} \{\|D\theta^n\|_T > M\}$$

and hence, since $\|D\theta^n\|_T$ is bounded in L^0:

$$P(\limsup_{n\to\infty} \|D\theta^n\|_T > M) = P\left(\bigcup_k \bigcap_{n\geq k} \{\|D\theta^n\|_T > M\}\right) \leq \sup_n P(\|D\theta^n\|_T > M) \leq \varepsilon$$

It follows that $\limsup_{n\to\infty} \|D\theta^n\|_T < \infty$ a.s. and hence $\sup_n \|D\theta^n\|_T < \infty$ a.s. Since $\theta_t^n \to \theta_t$ a.s., the thesis follows by Lebesgue dominated convergence theorem. $\square$

5.2.1 Continuous processes

Here we consider the case where X is a continuous process. In this setting, the semicontinuity of V_t^c can be shown using only arguments of duality between continuous functions and Radon measures.

Proposition 5.10. *If $\theta^n \in \mathcal{A}_c^k$, conv$(\{\|D\theta^i\|_T\}_{i\in\mathbb{N}})$ is bounded in L^0 and $\theta_t^n \to \theta_t$ a.s. in $dtdP$, then we have:*

$$V_T^c(\theta) \geq \limsup_{n\to\infty} V_T^c(\theta^n) \qquad \textit{for a.e. } \omega \tag{5.1}$$

Proof. By lemma 5.9, we have that $D\theta_t^n \rightharpoonup D\theta_t$, and hence, integrating by parts:

$$\begin{aligned}(\theta\cdot X)_T &= X_T\theta_T - X_0\theta_{0^+} - \int_{(0,T]} X_t dD\theta_t = \\ &= \lim_{n\to\infty}\left(X_T\theta_T^n - X_0\theta_{0^+}^n - \int_{(0,T]} X_t dD\theta_t^n\right) = \lim_{n\to\infty}(\theta^n\cdot X)_T\end{aligned}$$

and by the semicontinuity of the variation, we have:

$$\int_{[0,T]} X_t\cdot d|D\theta|_t \leq \liminf_{n\to\infty}\int_{[0,T]} X_t\cdot d|D\theta^n|_t$$

which completes the proof. $\square$

From the upper semicontinuity of V_T^c we easily obtain the lower semicontinuity of F:

Lemma 5.11. *Let ρ be a convex decreasing functional, H a $\mathcal{F}_T$-measurable random variable and $c > 0$. Denoting by $F : \theta \mapsto \rho(V_T^c(\theta) - H)$, we have:*

i) F is convex;

ii) F is lower semicontinuous with respect to $dtdP$-a.s. convergence.

Proof. Follows by a convexity argument and by Fatou's Lemma, exactly as in chapter 3, Lemma 4.3. □

The existence result is then an easy corollary:

Proposition 5.12. *Let X be a continuous process satisfying Definition 5.1 and Assumption 1.10. If ρ is a convex decreasing functional, then the problem:*

$$\min_{\theta \in \mathcal{A}_c^k} \rho\left(V_T^c(\theta) - H\right)$$

admits a solution.

Proof. Let $\theta^n \in \mathcal{A}_c^k$ be a minimizing sequence. From Proposition 5.4 we obtain a sequence $\eta^n \in \mathrm{conv}(\theta_n, \theta_{n+1}, \dots)$ such that $\eta^n \to \theta \in \mathcal{A}_c^k$ c.a.s. By the semicontinuity of ρ (Lemma 5.11), it follows that θ is a minimizer. □

5.2.2 The quasi-left continuous case

We now come to the more general case where X is a quasi-left continuous process on the whole interval $[0, T]$. Note that this assumption implies the continuity of X at T.

In this setting, arguments of duality between continuous functions and signed measures cannot be applied directly to show that the portfolio value is lower semicontinuous, as the discontinuities of X may be relevant for the limit measure $D\theta$. A simple example is given by a limit strategy which changes immediately after a jump has occurred. The idea of the next proof is that if the jumps of X are totally inaccessible, then the cases where convergence does not hold are negligible, as all strategies must be predictable.

Proposition 5.13. *If $\theta^n \in \mathcal{A}_c^k$, $\theta_t^n \to \theta_t$ c.a.s. and $conv(\{\|D\theta^i\|_T\}_{i\in\mathbb{N}})$ is bounded in L^0, then up to a subsequence:*

$$V_T^c(\theta) \geq \limsup_{n\to\infty} V_T^c(\theta^n) \qquad \text{for a.e. } \omega \tag{5.2}$$

We break the proof of Proposition 5.13 into two lemmas:

Lemma 5.14. *If $D\theta^n \rightharpoonup D\theta$ a.s., then we have:*

$$\int_{[0,T]} X_s \cdot d|D\theta|_s \leq \liminf_{n\to\infty} \int_{[0,T]} X_s \cdot d|D\theta^n|_s \qquad \text{for a.e. } \omega$$

In addition, if $\|D\theta^n\|_T \to \|D\theta\|_T$ a.s. (i.e. $D\theta^n$ converges to $D\theta$ in variation), then we obtain:

$$\int_{[0,T]} X_s \cdot d|D\theta|_s = \lim_{n\to\infty} \int_{[0,T]} X_s \cdot d|D\theta^n|_s \qquad \text{for a.e. } \omega$$

Proof. By a change of variable, we have:

$$\int_{[0,T]} X_s \cdot d|D\theta|_s = \int_0^\infty |D\theta|_T(X > x)dx$$

therefore it is sufficient to check that:

$$|D\theta|(X > x) \leq \liminf_{n\to\infty} |D\theta^n|(X > x) \tag{5.3}$$

Of course, the problem here is that the set $\{X > x\}$ is not necessarily open, as X may have discontinuities. However, X has only totally inaccessible jumps, therefore $\{\Delta X \neq 0\} = \bigcup_k [\![\sigma_k]\!]$ a.s., where σ_k is a sequence of totally inaccessible stopping times.

We denote by $\tau_k = \inf\{t \geq \sigma_k : t > x\}$, and define recursively:

$$\begin{aligned} &\tilde{\sigma}_1 = \sigma_1 && A_1 = [\![\sigma_1, \tau_1[\![\\ &\tilde{\sigma}_k = \sigma_k\big|_{\{\sigma_k \notin A_{k-1}\}} && A_k = A_{k-1} \cup [\![\sigma_k, \tau_k[\![\end{aligned}$$

where $\tau\big|_A = \tau 1_A + \infty 1_{\Omega\setminus A}$. It is easy to see that the set $\{\sigma_k \notin A_{k-1}\}$ is $\mathcal{F}_{\sigma_k}$-measurable, and hence $\tilde{\sigma}_k$ is a stopping time for all k.

Outside the random set $A_\infty = \bigcup_{k\geq 1} A_k$, the process X is continuous, therefore we can write

$$\{X > x\} = \bigcup_{k\in\mathbb{N}} [\![\tilde{\sigma}_k, \tau_k[\![\cup \bigcup_{k\in\mathbb{N}} (\alpha_k, \beta_k)$$

where $\{\alpha_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}}$ are $\mathcal{F}$-measurable random variables, and the union is disjoint by construction of $\tilde{\sigma}_k$ and α_k, β_k (we stick to the convention that $[a, b)$ is empty if $b \leq a$).

For each open interval (α_k, β_k), we obviously have:

$$|D\theta|(\alpha_k, \beta_k) \leq \liminf_{n\to\infty} |D\theta^n|(\alpha_k, \beta_k)$$

so it suffices to show that the same property holds for the stochastic intervals $[\![\sigma_k, \tau_k[\![$.

Up to a subsequence, we can assume that $|D\theta^n| \rightharpoonup \mu$, where $\mu \geq |D\theta|$. We define the predictable process:

$$\delta_t = \liminf_{n\to\infty} |D\theta^n|(0, t) - \mu(0, t)$$

Since we have that

$$\mu[0, t] \geq \limsup_{n\to\infty} |D\theta^n|[0, t] \geq \liminf_{n\to\infty} |D\theta^n|(0, t) \geq \mu(0, t)$$

it follows that $0 \leq \delta_t \leq \mu\{t\}$ and hence $\delta_t > 0$ for at most countably many t.

As a result (see Dellacherie and Meyer [DM78], Chapter IV, Theorem 88), $\{(t, \omega) : \delta_t > 0\} = \bigcup_k [\![\pi_k]\!]$, where $\{\pi_k\}_{k\in\mathbb{N}}$ is a sequence of predictable stopping times. It follows that $P(\pi_j = \sigma_k) = 0$ for all j, k, and hence $\lim_{n\to\infty} |D\theta^n|(]\!]0, \sigma_k[\![) = \mu(]\!]0, \sigma_k[\![)$ a.s.

We have that:

$$\mu(]\!]0, \tau_k[\![) \leq \liminf_{n\to\infty} |D\theta^n|(]\!]0, \tau_k[\![) = \lim_{n\to\infty} |D\theta^n|(]\!]0, \sigma_k[\![) + \liminf_{n\to\infty} |D\theta^n|([\![\sigma_k, \tau_k[\![) =$$
$$= \mu(]\!]0, \sigma_k[\![) + \liminf_{n\to\infty} |D\theta^n|([\![\sigma_k, \tau_k[\![)$$

whence:

$$\mu([\![\sigma_k, \tau_k[\![) \leq \lim_{n\to\infty} |D\theta^n|[\![\sigma_k, \tau_k[\![$$

which completes the proof. □

Lemma 5.15. *If $\theta^n_t \to \theta_t$ c.a.s. and conv$(\{\|D\theta^i\|_T\}_{i\in\mathbb{N}})$ is bounded in L^0, then up to a subsequence $(\theta^n \cdot X)_T$ converges in probability to $(\theta \cdot X)_T$.*

Proof. We have that:

$$
\begin{aligned}
(\theta^n \cdot X)_T = X_T\theta^n_T - X_0\theta^n_{0+} - \int_{(0,T]} X_t dD\theta^n_t = \\
= X_T\theta^n_T - X_0\theta^n_{0+} - \int_{(0,T]} X_t d(D\theta^n)^+_t - \int_{(0,T]} X_t d(D\theta^n)^-_t
\end{aligned}
$$

where $(D\theta^n)^+$ and $(D\theta^n)^-$ denote respectively the positive and negative parts in the Hahn decomposition of $D\theta^n$. Up to subsequences, we can assume that $|D\theta^n|_T$ converges a.s. and hence that $(D\theta^n)^+ \rightharpoonup \nu^+$ and $(D\theta^n)^- \rightharpoonup \nu^-$, where ν^+ and ν^- are positive vector measures. Applying Lemma 5.14 to the last two integrals above, we obtain that:

$$
\begin{aligned}
\lim_{n\to\infty} (\theta^n \cdot X)_T = X_T\theta_T - \int_{[0,T]} X_t d\nu^+_t - \int_{[0,T]} X_t d\nu^-_t = \\
= X_T\theta_T - \int_{[0,T]} X_t dD\theta_t = (\theta \cdot X)_T
\end{aligned}
$$

□

Proof of Proposition 5.13. The thesis follows from Lemmas 5.9, 5.15 and 5.14. □

As in the continuous case, the existence of minimizers is easily obtained:

Theorem 5.16. *Let X be a quasi-left continuous process satisfying Definition 5.1 and Assumption 1.10. If ρ is a convex decreasing functional, then the problem:*

$$\min_{\theta\in\mathcal{A}^k_c} \rho\left(V^c_T(\theta) - H\right)$$

admits a solution.

Proof. As in Proposition 5.12, it follows from Proposition 5.4 and 5.13. □

5.2.3 The semimartingale case

In the previous section, we have shown how the almost sure convergence of integrands implies the convergence of elementary stochastic integrals. Since the integrating process X was not necessarily a semimartingale, the general results of stochastic integration were not available, and we had to rely on the basic properties of paths.

In particular, the proof of Lemma 5.15 needed the second part of 5.14, through an integration by parts argument. In the case where X is a locally integrable semimartingale, we can expect a simplification, and here we show an alternative proof, based on standard stochastic integration theory.

This proof has a drawback, however. In fact, in the previous proof the integrands θ^n only needed to converge almost surely in $dtdP$. By contrast, the next proof needs the full result of Proposition 5.4, where convergence is obtained in the (generally stronger) sense on Definition 5.2.

Lemma 5.17. *Let X be a locally integrable semimartingale. If $\theta^n_t \to \theta_t$ c.a.s. and conv$(\{\|D\theta^i\|_T\}_{i\in\mathbb{N}})$ is bounded in L^0, then $(\theta^n \cdot X)_T$ converges in probability to $(\theta \cdot X)_T$.*

Proof. With no loss of generality, we can assume that $\theta^n_t \to 0$ c.a.s. Let $X = M + A$ the Doob-Meyer decomposition of X. Since X is quasi left-continuous, A has continuous paths. We have that:

$$|(\theta^n \cdot X)_t| \leq |(\theta^n \cdot A)_t| + |(\theta^n \cdot M)_t|$$

and as in Proposition 5.10 we obtain that $(\theta^n \cdot A)$ converges in probability to zero. It remains to show that the same holds for $(\theta^n \cdot M)$.

Up to a sequence of convex combinations, we can assume that $\sup_n |D\theta^n|_t = V_t(\omega) < \infty$ a.s., and V_t is left-continuous. We have:

$$|(\theta^n \cdot M)_t| = \left|\theta^n_t M_t - \int_{[0,t]} M_s d\theta^n_s\right| \leq 2|D\theta^n_t| M^*_t \leq 2V_t M^*_t$$

Let ρ_k be a sequence of stopping times such that $M^{\rho_k}_t$ is integrable. Denoting by $\sigma_k = \inf\{t : V_t \geq k\}$ and $\tau_k = \rho_k \wedge \sigma_k$, we obtain:

$$E\left[|(\theta^n \cdot M)^{\tau_k}_T|^2\right] = E\left[\int_{[0,T\wedge\tau_k]} |\theta^n_s|^2 d\langle M\rangle_s\right]$$

Since $\langle M\rangle$ is continuous, the measure $d\langle M\rangle$ is diffuse, and for a.e. ω the countable set $\{t : \theta^n_t \not\to \theta_t\}$ is $d\langle M\rangle$-negligible.

Since $|\theta^n_s| \le V_{\tau_k} \le k$, by dominated convergence it follows that $(\theta^n \cdot M)^{\tau_k}_T \to 0$ in L^2, and hence in probability. Finally, we have that:

$$\begin{aligned} |(\theta^n \cdot M)_T| \le & |(\theta^n \cdot M)_T - (\theta^n \cdot M)^{\tau_k}_T| + |(\theta^n \cdot M)^{\tau_k}_T| \le \\ \le & |D\theta^n|[\tau_k \wedge T, T] M^*_T + (|\theta^n_{\tau_k}| + |\theta^n_T|) M^*_T + |(\theta^n \cdot M)^{\tau_k}_T| \end{aligned}$$

Since $\sup_n |D\theta^n|_t \le V_t(\omega)$, the first term in the right hand side can be made arbitrarily small, uniformly in n. The other terms converge to zero as $n \to \infty$, and the thesis follows. □

5.3 Constrained problems

We now see how the results in the previous section can be extended so as to allow for constraints on strategies. As in chapter 3, this generalization poses no particular problems, as long as the constraints are closed, and remain stable with respect to the transformations on minimizing strategies.

Since the relative compactness of minimizing sequences depends on the key Lemma 5.6, we are able to consider closed convex constraints considered in Chapter 3

The existence result for constrained problems can be formulated as follows:

Proposition 5.18. *Let K be a convex constraint. Denoting by:*

$$\mathcal{A}^k_c(K) = \{\theta_t \in \mathcal{A}^k_c : \theta \textit{ satisfies } K\}$$

If $\mathcal{A}^k_c(K)$ is not empty, then the minimum problem

$$\min_{\theta \in \mathcal{A}^k_c(m,M)} \rho\left(V^c_T(\theta) - H(\theta)\right)$$

admits a solution.

Proof. Analogous to that of Proposition 3.23. □

5.4 Utility maximization

It is natural to embed the *utility maximization* problem in the framework of Proposition 5.12 by choosing $\rho(X) = E[-U(X)]$. However, while conditions *i*) and *ii*) in Definition 1.14 clearly hold for any utility function U, the Fatou property *iii*) is trivially satisfied only when U is bounded.

In the frictionless case, Kramkov and Schachermayer [KS99] have shown that a solution to the utility maximization problem exists if and only if U has *reasonable asymptotic elasticity*, defined as follows:

Definition 5.19. *The* asymptotic elasticity *of an increasing concave function U is defined by:*

$$AE_{+\infty}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}$$

and U has reasonable asymptotic elasticity *if $AE_{+\infty}(U) < 1$.*

In practice, for twice differentiable utility functions this condition is equivalent to the requirement:

$$\lim_{x\to+\infty} -\frac{xU''(x)}{U'(x)} > 0 \tag{5.4}$$

as it can be easily checked by a straightforward application of De L'Hôpital's rule. This condition has a clear economic interpretation, since $-\frac{xU''(x)}{U'(x)}$ represents the relative risk aversion at the level of wealth x. Hence, (5.4) prescribes that the agent keeps a certain positive relative risk aversion for arbitrarily large levels of wealth.

In the case of transaction costs, existence of solutions can be established under the same conditions:

Theorem 5.20. *Let X be a quasi-left continuous process satisfying Definition 5.1 and Assumption 1.10. Let $U : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be an increasing and convex function, with $AE(U)_{+\infty} < 1$.*

If $\max_{\theta\in\mathcal{A}_c^k} E\left[U\left(V_T^c(\theta)\right)\right] < \infty$, then the utility maximization problem

$$\max_{\theta\in\mathcal{A}_c^k} E\left[U\left(V_T^c(\theta)\right)\right] \tag{5.5}$$

admits a solution.

The proof of this result can be obtained from its frictionless counterpart: here we show how the proof of Schachermayer [Sch01b] can be easily adapted to this setting.

We need the following lemma:

Lemma 5.21 (Schachermayer, [Sch01b]). *Let* $(f_n)_{n=1}^{\infty} \geq 0$ *be random variables on* $(\Omega, \mathcal{F}, P)$ *converging a.s. to* f_0*. Suppose that* $\lim_{n\to\infty} E[f_n] = E[f_0] + \alpha$*, for some* $\alpha > 0$*. Then for all* $\varepsilon > 0$ *there exist* $n, m > \varepsilon^{-1}$ *and disjoint sets* A_n, A_m *such that the following conditions are satisfied:*

i) $f_n \geq \varepsilon^{-1}$ *on* A_n *and* $f_m \geq \varepsilon^{-1}$ *on* A_m

ii) $E[f_n 1_{A_n}] > \alpha - \varepsilon$ *and* $E[f_m 1_{A_m}] > \alpha - \varepsilon$

iii) $E\left[f_n 1_{\Omega \setminus (A_n \cup A_m)}\right] > E[f_0] - \varepsilon$ *and* $E\left[f_m 1_{\Omega \setminus (A_n \cup A_m)}\right] > E[f_0] - \varepsilon$

Proof of Theorem 5.20. Let θ^k be a maximizing sequence for (5.5). Since $AE_{+\infty}(U) < 1$, by Lemma 6.3 in [KS99] there exists some $\beta > 1$ such that $U(\frac{x}{2}) > \frac{\beta}{2} U(x)$ for all $x \geq x_0$.

Since X is compact arbitrage free, by Proposition 5.4 we can assume up to a sequence of convex combinations that $\theta^k \to \theta \in \mathcal{A}_c^k$ a.s. in $dtdP$. By Proposition 5.13, we have that:

$$V_T^c(\theta) \geq \limsup_{k\to\infty} V_T^c(\theta^k) \qquad \text{a.s. in } P$$

We need to show that $\lim_{k\to\infty} E\left[U(V_T^c(\theta^k))\right] \leq E[U(V_T^c(\theta))]$. By contradiction, suppose that:

$$\lim_{n\to\infty} E[U(V_T^c(\theta^n))] - E[U(V_T^c(\theta^n))] = \alpha > 0$$

Then, by Lemma 5.21 we could find n, m arbitrarily large and A_n, A_m such that:

$$E\left[U\left(\frac{V_T^c(\theta^n) + V_T^c(\theta^m)}{2}\right)\right] = E\left[U\left(\frac{V_T^c(\theta^n) + V_T^c(\theta^m)}{2}\right) 1_{\Omega\setminus(A_n\cup A_m)}\right] + \\ + E\left[U\left(\frac{V_T^c(\theta^n) + V_T^c(\theta^m)}{2}\right) 1_{A_n\cup A_m}\right]$$

By the condition $AE_{+\infty}(U) < 1$, for the second term we have:

$$E\left[U\left(\frac{V_T^c(\theta^n) + V_T^c(\theta^m)}{2}\right) 1_{A_n\cup A_m}\right] \geq \beta E\left[\frac{U(V_T^c(\theta^n) + V_T^c(\theta^m))}{2} 1_{A_n\cup A_m}\right] \geq \\ \geq \frac{\beta}{2}\left(E\left[U\left(V_T^c(\theta^n)\right) 1_{A_n}\right] + E\left[U\left(V_T^c(\theta^m)\right) 1_{A_m}\right]\right) \geq \beta(\alpha - \varepsilon)$$

while for the first term:

$$E\left[U\left(\frac{V_T^c(\theta^n)+V_T^c(\theta^m)}{2}\right)1_{\Omega\setminus(A_n\cup A_m)}\right] \geq$$
$$\geq \frac{1}{2}\left(E\left[U(V_T^c(\theta^n))1_{\Omega\setminus(A_n\cup A_m)}\right]+E\left[U(V_T^c(\theta^m))1_{\Omega\setminus(A_n\cup A_m)}\right]\right) \geq$$
$$\geq E\left[U(V_T^c(\theta))\right]-\varepsilon$$

and hence:

$$E\left[U\left(\frac{V_T^c(\theta^n)+V_T^c(\theta^m)}{2}\right)\right] \geq E\left[U(V_T^c(\theta))\right]+\alpha+((\beta-1)\alpha-\varepsilon(\beta+1))$$

Since ε can be chosen arbitrarily small, we can assume that the last term in the right is positive, but this leads to a contradiction, since $V_T^c(\theta^n)$ was a maximizing sequence, and $\lim_{n\to\infty} E\left[U(X_n)\right]=E\left[U(V_T^c(\theta))\right]+\alpha$ was the supremum. □

5.5 The edge of the no-arbitrage condition

The next counterexample shows that the compactness result of Proposition 5.4 does not hold when $\gamma=k$, despite the market remains arbitrage free.

Example 5.22. Let R_t be a Brownian Motion started in 1 and reflected between $\frac{1}{1+\varepsilon}$ and $\frac{1}{1-\varepsilon}$, and denote $X_t=R_{\frac{t}{T-t}}$.

Note that $1-\varepsilon \leq \frac{1}{X_t} \leq 1+\varepsilon$, and by Proposition 2.2 the asset X is arbitrage-free for $k \geq \varepsilon$ (since $\tilde{X}_t \equiv 1$ is obviously a martingale under $Q=P$).

Consider the stopping times $\{\tau_i\}_{i\geq 1}$ and $\{\sigma_i\}_{i\geq 0}$ defined as follows:

$$\begin{cases} \sigma_0 & =0 \\ \tau_{i+1} & =\inf\{s>\sigma_i : X_s=\frac{1}{1+\varepsilon}\} \\ \sigma_i & =\inf\{s>\tau_i : X_s=\frac{1}{1-\varepsilon}\} \end{cases}$$

and the strategy θ defined by:

$$\theta_t=\begin{cases} 0 & \text{for } t\in(\sigma_k,\tau_{k+1}] \\ \delta & \text{for } t\in(\tau_k,\sigma_k] \end{cases}$$

Note that by construction $\sigma_i, \tau_i < \infty$ a.s. for all i.

Consider constant proportional transaction costs $k_t = k$. For $k = \varepsilon$, it is easy to see that $V_t^c(\theta) \in [c - \delta\frac{k}{1+\varepsilon}, c + \delta\frac{k}{1-\varepsilon}]$, and hence θ is admissible for some small $\delta > 0$. Also, $|D\theta|_{\sigma_i} = 2\delta i$, hence $|D\theta|_T = \infty$ a.s.

Defining $\theta^n = \theta^{\sigma_n}$ (that is, θ stopped at σ_n), we obtain that $\{|D\theta^n|_T\}_n = \{2\delta n\}_n$ is not bounded in L^0, therefore the assumptions of Lemma 5.6 are not satisfied. Also, all sequences of convex combinations of θ^n converge to θ a.s., hence there is no hope that one of them converges to a function of finite variation.

In practice, the asset X in the above example allows a trivial arbitrage strategy: buy at $\frac{1}{1+\varepsilon}$, sell at $\frac{1}{1-\varepsilon}$. However, with transaction costs, this strategy remains an arbitrage depending on the cost size: for $k < \varepsilon$, it still yields a profit, while for $k > \varepsilon$ it leads to a net loss (hence θ is not admissible). At the critical value $k = \varepsilon$, the trading gain is exactly offset by the transaction cost, and the portfolio value remains bounded though the trading strategy becomes more and more hectic as $t \to T$.

5.6 Intertemporal Problems

In this chapter we prove the existence of solutions to intertemporal optimization problems with market frictions.

The main difference from the fixed horizon case is the presence in this setting of consumption as an additional control variable. However, such difference is rather irrelevant in terms of technical difficulties, as cumulative consumption is modeled as an increasing process C_t, hence of finite variation. This fact, together with the observation that consumption is bounded by the budget constraint, allows to employ in the space of consumption plans exactly the same compactness tools developed for trading strategies. This means that the results in this setting are almost straightforward consequences of those established in the fixed horizon case, although the underlying economic problems are very different.

Definition 5.23. *For any $c, k > 0$, we define the space of* admissible consumption/investment plans *as:*

$$\mathcal{R}_c^k = \{(\theta, C) : \theta \in \mathcal{A}_c^k, C \in \mathcal{C}, V_T^c(\theta, C) \geq 0\}$$

Proposition 5.24. *Let X be a quasi-left continuous process satisfying Assumption 1.10 and Definition 5.1. If ρ is an intertemporal convex decreasing functional, then the problem:*

$$\min_{(\theta,C)\in\mathcal{R}_c^k} \rho\,(C)$$

admits a solution.

Proof. Let (θ^n, C^n) a minimizing sequence. By Proposition 5.4, up to a sequence of convex combinations we can assume that θ^n converges a.s. in $dtdP$ to some finite-variation predictable finite process θ.

Also, by the budget constraint $V_T(\theta, C) \geq 0$ we have:

$$C_T \leq c + (\theta \cdot X)_t - \int_{[0,t]} k_s X_s \cdot d|D\theta|_s - k_t X_t \cdot |\theta_t|$$

By the assumption that X is γ-arbitrage free, it follows that the right-hand side is bounded in L^0, uniformly in θ. As a result, for any minimizing sequence C^n we have that $\{C_T^n\}_n$ is bounded in L^0. Hence, in the same fashion as in Lemma 5.6 (or simply applying Lemma 4.2 in [Kra96]), we obtain, up to a sequence of convex combinations, that C^n converges a.s. in $dtdP$ to some $C \in \mathcal{C}$.

It only remains to prove that the strategy (θ, C) is optimal and satisfies the budget constraint. For the latter, we note that:

$$V_t(\theta, C) = V_t(\theta) - C_t$$

The first term in the right-hand side is upper semicontinuous by Lemma 3.8. The consumption term must be lower semicontinuous, since a.s. convergence of C_t^n readily implies the weak star convergence of the measures dC^n. As a result, the whole quantity $V_t(\theta, C)$ is upper semicontinuous, and the budget constraints is preserved in the limit.

The proof is complete observing that, by the convexity of ρ, convex combinations of minimizing sequences are themselves minimizing, and the Fatou property of ρ guarantees the optimality of the limit (θ, C). $\square$

Example 5.25 *(Hindy-Huang-Kreps [HHK92] preferences).* These preferences are obtained by risk-functionals of the form:

$$\rho(C) = -E\left[\int_{[0,T]} U(t, Y_t(C))dt\right]$$

where $U : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function, increasing and strictly concave in the second argument. The function U measures the "felicity" of the agent at time t, taking into account present as well as past consumption through the habit formation function Y_t, defined as follows:

$$Y_t(C) = \eta e^{-\beta t} + \int_{[0,T]} \beta e^{-\beta(t-s)} dC_s$$

The parameters $\eta, \beta > 0$ represent respectively the initial level of satisfaction, and the time decay of past consumption as factor of current satisfaction. Note that for $\eta = 0$ and in the limit $\beta \to \infty$ we obtain $Y_t(C) = C_t$.

Remark 5.26. The most classical case in microeconomics is probably that of time-additive preferences, used among others by Merton [Mer69]. In this case, the risk functional takes the following form:

$$\rho(C) = -E\left[\int_{[0,T]} e^{-rt} U(C'_t)dt + B(C_T)\right]$$

where r is an impatience factor, U a utility function depending on the local consumption rate C'_t and B a bequest valuation function depending on the final consumption gulp C_T.

In this case, the functional is defined only on absolute continuous consumption plans. This means that our result cannot be directly applied to this setting, as it provides a candidate optimal consuption plan which does not necessarily belong to this class.

5.7 Notes

The problems addressed in this chapter have recently been studied by several authors, under different model assumptions.

Kamizono [Kam01] has obtained existence results for hedging problems under transaction costs in the continuous semimartingale case, within the framework introduced by Kabanov et al. [KRS01]. In the same setting, he has proved the existence of solutions for utility maximization problems, both from terminal wealth and from intertemporal consumption.

Utility maximization problems under transaction costs have also been studied by Deelstra, Pham and Touzi [DPT00] in the general semimartingale case, with a particular attention to the case of nonsmooth utility functions, which requires the use of nonsmooth convex analysis tools.

Appendix A

We recall here a few results in functional analysis and stochastic integration that we use in the main text. Here T denotes a generic stopping time.

This result dates back to Banach, and is well-known:

Theorem A.1. *Let x_n be a relatively weakly compact sequence in a Banach space V. Then there exists a sequence of convex combinations $y_n = \sum_{i=n}^{\infty} \alpha_i^n x_n$ such that y_n converges in the norm topology of V.*

Bounded sets are relatively compact in L^p spaces for $p > 1$. For L^1 this is not the case, since its weak star closure leads to the space of Radon measures. Nonetheless, relative compactness can be recovered through convex combinations, as shown by the following:

Theorem A.2 (Komlós). *Let X_n be a sequence of random variables, such that $\sup_n E|X_n| < \infty$. Then there exists a subsequence X'_n and a random variable $X \in L^1$ such that, for each subsequence X''_n of X'_n,*

$$\frac{1}{n}\sum_{i=1}^{n} X''_n \to X \ \ a.s.$$

The next type of results states when a sequence of stochastic integrals converges to a stochastic integral. Again, the situation is different for $p > 1$ and $p = 1$. The first case is a classic result of stochastic integration, and dates back to Kunita-Watanabe for $p = 2$:

Theorem A.3 (Kunita-Watanabe). *Let X be a continuous local martingale, θ^n a sequence of X-integrable predictable stochastic processes such* *that each* $\int_0^t \theta_s^n dX_s$ *is a* L^p*-bounded martingale, and such that the sequence* $\int_0^\infty \theta_s^n dX_s$ *converges to a random variable* G *in the norm topology of* L^p*.*

Then there is an $\mathcal{F}^X$*-predictable stochastic process* θ *such that* $\int_0^t \theta_s dX_s$ *is an* L^p*-bounded martingale, and such that* $\int_0^t \theta_s dX_s = G$*.*

The case $p = 1$ is due to Yor [Yor78]:

Theorem A.4 (Yor). *Let* X *be a continuous local martingale,* θ^n *a sequence of* X*-integrable predictable stochastic processes such that each* $\int_0^t \theta_s^n dX_s$ *is a uniformly integrable martingale, and such that the sequence* $\int_0^\infty \theta_s^n dX_s$ *converges to a random variable* G *in the norm topology of* L^1 *(or even in the* $\sigma(L^1, L^\infty)$ *topology).*

Then there is an $\mathcal{F}^X$*-predictable stochastic process* θ *such that* $\int_0^t \theta_s dX_s$ *is a uniformly integrable martingale, and such that* $\int_0^t \theta_s dX_s = G$*.*

The main difference between $p > 1$ and $p = 1$ is that in the latter case the norms $L^p : M \mapsto E\left[M_\infty^p\right]$ and $\mathcal{H}^p : M \mapsto \langle M \rangle_\infty^{\frac{p}{2}}$ are no longer equivalent. Yor's idea is to reduce the L^1 case to $\mathcal{H}^1$ by stopping arguments.

In fact, Theorem A.4 is a consequence of a more general result (see Yor [Yor78], Theorem 2.4 page 277), which gives further information on converging sequences of uniformly integrable martingales. We report this result with its proof, and an easy corollary used in this paper.

Note that a different proof of A.4 can be found in Delbaen and Schachermayer [DS99], with an excellent exposition of the properties of the space $\mathcal{H}^1$.

Theorem A.5 (Yor). *Let* A *be a set of uniformly integrable martingales, and denote*

$$\Phi(A) = \{Y^T : \ Y \in A, \ Y^T \in \mathcal{H}^1\}$$

Let Y *be a uniformly integrable martingale,* $Y^n \in A$ *and* $Y^n \to Y$ *in* L^1 *(or even weakly in* $\sigma(L^1, L^\infty)$*). Then, for all stopping times* T *such that* $Y^T \in \mathcal{H}^1$*, we have that* Y^T *belongs to the closure of* $\Phi(A)$ *in the weak topology* $\sigma(H^1, BMO)$*.*

Proof. We separate the proof into three steps.

Step 1: we first show that if the theorem holds for $Y \in \mathcal{H}^1$ and $T = \infty$, then it holds in general. If $Y^n \rightharpoonup Y$ in $\sigma(L^1, L^\infty)$, then $(Y^n)^T \rightharpoonup Y^T$, and we can apply the theorem to $(Y^n)^T, Y^T$ (since $Y^T \in \mathcal{H}^1$ by assumption) obtaining that Y^T belongs to the closure of $\Phi(\{(Y^n)^T\}_n)$, which is smaller than $\Phi(A)$.

Step 2: we further reduce the proof to the case of $Y^n \in \mathcal{H}^1$. For all n, there exists a sequence of stopping times $S^n_k \to \infty$ such that $(Y^n)^{S^n_k} \in \mathcal{H}^1$. Denoting by $Z_n = (Y^n)^{S^n_k}$, by martingale convergence there exists some $k = k_n$ such that $\|Y^n_\infty - Z^n_\infty\|_{L^1} = \|Y^n_\infty - E\left[Y^n_\infty | \mathcal{F}_{S^n_k}\right]\|_{L^1} \leq \frac{1}{n}$. Therefore, $Z^n \in \mathcal{H}^1$ for all n, and we have, for all $g \in L^\infty$:

$$|E\left[(Z^n_\infty - Y_\infty)g\right]| \leq \|Z^n_\infty - Y^n_\infty\|_{L^1}\|g\|_{L^\infty} + |E\left[(Y^n_\infty - Y_\infty)g\right]|$$

therefore Z^n_∞ converges weakly in $\sigma(L^1, L^\infty)$ to Y_∞. Also, the set $\Phi(\{Z^n\}_n)$ is smaller then $\Phi(A)$.

Step 3: we now give the proof under the assumptions $Y^n, Y \in \mathcal{H}^1, T = \infty$. It is sufficient to show, for any finite subset $(U^1, \ldots, U^d) \in BMO$, that there exists some stopping time T such that:

$$|E\left[[(Y^n)^T - Y, U^i]_\infty\right]| < \varepsilon \quad \text{for all } i$$

We take $T = \inf\{t : \sum_{i=1}^d |U^i| \geq k\}$, choosing k such that:

$$E\left[\int_{(T,\infty]} |d[Y, U^i]_s|\right] < \frac{\varepsilon}{2} \quad \text{for all } i$$

which is always possible by the Fefferman inequality, since $[Y, U^i]$ has integrable variation, $Y \in \mathcal{H}^1$, and $U^i \in BMO$. Therefore it remains to show that, for some fixed T, and for all $i = 1, \ldots, d$:

$$\lim_{n\to\infty} E\left[[(Y^n)^T, U^i]_\infty\right] = \lim_{n\to\infty} E\left[[Y^n, (U^i)^T]_\infty\right] = E\left[[Y, (U^i)^T]_\infty\right]$$

U^i is bounded in $[0, T)$, but it belongs to BMO, it has bounded jumps, hence it is also bounded on $[0, T]$. As a result, $(U^i)^T$ is bounded. The local martingale $[Y^n, (U^i)^T] - Y^n(U^i)^T$ hence belongs to the class D, and we have:

$$E\left[[Y^n, (U^i)^T]_\infty\right] = E\left[Y^n_\infty U^i_T\right] \quad \text{and} \quad E\left[[Y, (U^i)^T]_\infty\right] = E\left[Y_\infty U^i_T\right]$$

Finally, $U^i_T \in L^\infty$, and the thesis follows from the assumption $Y^n \rightharpoonup Y$ in $\sigma(L^1, L^\infty)$. $\square$

Remark A.6. The statement of Theorem A.5 with $A = \{Y^n\}_n$ says that, for any $Y^\tau \in \mathcal{H}^1$, there exists a sequence of indices n_k, and a sequence of stopping times τ_k such that $Y_{n_k}^{\tau_k} \rightharpoonup Y^\tau$ in $\sigma(H^1, BMO)$. However, *a priori* the sequence n_k may not tend to infinity, and the the stopping times τ_k may not converge almost surely to τ.

The proof provides more information on these issues. For the first, note that in fact $n_k = k$. For the latter, let us look more closely to the three steps.

Step 1 simply shows that τ_k may be chosen such that $\tau_k \leq \tau$ a.s.

In Step 2, we have $S_k^n \to \infty$, and k_n must be sufficiently high. Therefore we can replace it with some higher k_n such that the condition $P(S_{k_n}^n < n) < \frac{1}{n}$ is satisfied as well.

Likewise, in Step 3 the stopping time T needs to be sufficiently high, hence it may be chosen to satisfy the condition $P(T < n) < \frac{1}{n}$. By a diagonalization argument, we can select a sequence $(Y^n)^{T_n}$ such that $(Y^n)^{T_n} \rightharpoonup Y$ in $\sigma(H^1, BMO)$, and $T_n \to \infty$ a.s.

The previous Remark provides the following:

Corollary A.7. *If $Y^n \rightharpoonup Y$ in $\sigma(L^1, L^\infty)$, and $Y^\tau \in \mathcal{H}^1$ for some stopping time τ, then there exists a subsequence n_k and a sequence of stopping times $\tau_k \to \tau$ a.s. such that $(Y^{n_k})^{\tau_k} \rightharpoonup Y^\tau$ in $\sigma(\mathcal{H}^1, BMO)$.*

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