ABSTRACT, CLASSIC, AND EXPLICIT TURNPIKES

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Portfolio turnpikes state that, as the investment horizon increases, optimal portfolios for generic utilities converge to those of isoelastic utilities. This paper proves three kinds of turnpikes. The *abstract* turnpike, valid in a general semimartingale setting, states that final payoffs and portfolios converge under their myopic probabilities. Diffusion models with several assets and a single state variable lead to the *classic* turnpike, in which optimal portfolios converge under the physical probability, and to the *explicit turnpike*, which identifies the limit of optimal portfolios in terms of the solution of an ergodic HJB equation.

1. Introduction. In the theory of portfolio choice, ruled by particular and complicated results, turnpike theorems are the happy exception – general and simple. Loosely defined, these theorems state that, when the investment horizon is distant, the optimal portfolio of *any* investor approaches that of an investor with isoelastic utility, suggesting that *for long-term investments, only isoelastic utilities matter*.

This paper proves turnpike theorems in a general framework, which includes discrete and continuous time, and nests diffusion models with several assets, stochastic drifts, volatilities, and interest rates. The paper departs from the existing literature, in which either asset returns are independent over time, or markets are complete. It is precisely when both these assumptions fail that portfolio choice becomes most challenging – and turnpike theorems most useful.

Our results have three broad implications. First, turnpike theorems are a powerful tool in portfolio choice, because they apply not only when optimal portfolios are myopic, but also when the *intertemporal hedging* component is present. Finding this component is the central problem of portfolio choice, and the only tractable but non trivial analysis is based on isoelastic utilities, combined with long horizon asymptotics. Turnpike theorems make this analysis relevant for a large class of utility functions, and for large but finite horizons.

Second, we clarify the roles of preferences and market structure for turnpike results. Under regularity conditions on utility functions, we show that an *abstract turnpike* theorem holds regardless of market structure, as long as utility maximization is well posed, and longer horizons lead to higher payoffs. This abstract turnpike yields the convergence of optimal portfolios to their isoelastic limit under the *myopic* probability $\mathbb{P}^T$, which changes with the horizon. Market structure becomes crucial to pass from from the abstract to the *classic turnpike* theorem, in which convergence holds under the physical probability $\mathbb{P}$.

Third, in addition to the classical version, we prove a new kind of result, the *explicit turnpike*, in which the limit portfolio is identified as the *long-run optimal* portfolio, that is the solution to an ergodic Hamilton Jacobi Bellman equation. This result offers the first theoretical basis to the long-standing practice of interpreting solutions of ergodic HJB equations as long-run limits of utility

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maximization problems[1]. We show that this intuition is indeed correct for a large class of diffusion models, and that its scope includes a broader class of utility functions.

Portfolio turnpikes start with the work of Mossin (1968) on affine risk tolerance ($-U'(x)/U''(x) = ax + b$), which envisions many of the later developments. In his concluding remarks, he writes: *"Do any of these results carry over to arbitrary utility functions? They seem reasonable enough, but the generalization does not appear easy to make. As one usually characterizes those problems one hasn't been able to solve oneself: this is a promising area for future research".*

Leland (1972) coins the expression *portfolio turnpike*, extending Mossin's result to larger classes of utilities, followed by Ross (1974) and Hakansson (1974). Huberman and Ross (1983) prove a necessary and sufficient condition for the turnpike property. As in the previous literature, they consider discrete time models with independent returns. Cox and Huang (1992) prove the first turnpike theorem in continuous time, using contingent claim methods. Jin (1998) extends their results to include consumption, and Huang and Zariphopoulou (1999) obtain similar results using viscosity solutions. Dybvig, Rogers, and Back (1999) dispose of the assumption of independent returns, proving a turnpike theorem for complete markets based on the Brownian filtration.

In summary, the literature either exploits independent returns, which make dynamic programming attractive, or complete markets, which make martingale methods convenient. Since market completeness and independence of returns have a tenuous relation, neither of these concepts appears to be central to turnpike theorems. We confirm this intuition, by relaxing both assumptions in this paper.

The main results are in section 2, which is divided into two parts. The first part shows the conditions leading to the abstract turnpike, whereby final payoffs and portfolios converge under the myopic probabilities $\mathbb{P}^T$. Regarding preferences (Assumption 2.1), we require a marginal utility that is asymptotically isoelastic as wealth increases (CONV), and that does not stray from its reference at low wealth levels ((LB-0) and (UB-0)). Market structure remains irrelevant, beyond the basic conditions that utility maximization is well-posed (Assumption 2.3), and that wealth processes can be freely compounded (Assumption 2.2).

The second part of section 2 states the classic and explicit turnpike theorems for a class of diffusion models with several assets, but with a single state variable driving expected returns, volatilities and interest rates. Under further well-posedness assumptions, we show a classic turnpike theorem, in which optimal portfolios of generic utility functions converge to their isoelastic counterparts. The same machinery leads to the explicit turnpike, in which optimal portfolios for a generic utility and a finite horizon converge to the long-run optimal portfolio, that is the solution of an ergodic HJB equation. The examples in Section 3 show the significance and the limits of these results, and in particular the relevance of the assumptions made to obtain them.

Section 4 contains the proofs of the abstract turnpike, while the classic and the explicit turnpike for diffusions are proved in section 5. Section 4 is divided into three parts. The first part proves for log utility a preliminary turnpike property, that is the convergence of the ratio of final payoffs. The second part of section 4 proves the same result for power utility, and is independent of the first part. In the third part, the convergence of wealth processes is derived from the convergence of payoff ratios. Section 5 is divided into three parts, the first one proving the necessary verification results, the second one obtaining the convergence of conditional densities that underlies the classic and explicit turnpikes, then these turnpike results are proved in the third part.

In conclusion, this paper shows that turnpike theorems are an useful tool to make portfolio choice tractable, even in the most intractable setting of incomplete markets combined with stochastic

[1]This interpretation underpins the literature on *risk-sensitive control*, introduced by Fleming and McEneaney (1995), and applied to optimal portfolio choice by Fleming and Sheu (2000; 2002), Bielecki et al. (2000; 2002), Nagai and Peng (2002b; 2002a) among others.

investment opportunities. Still, these results are likely to admit extensions to more general settings, like diffusions with multiple state variables, and processes with jumps. As Mossin put it gracefully, this is a promising area for future research.

2. Main Results. This section contains the statements of the main results and their implications. The first subsection states an abstract version of the turnpike theorem, which focuses on payoff spaces and wealth processes, without explicit reference to the structure of the underlying market. In this setting, asymptotic conditions on the utility functions and on wealth growth imply that, as the horizon increases, optimal wealths and optimal portfolios converge to their isoelastic counterparts.

The defining feature of the abstract turnpike is that convergence takes place under a family of *myopic* probability measures that change with the horizon. By contrast, in the *classic* turnpike the convergence of these quantities holds under the physical probability measure. Thus, passing from the abstract to the classic turnpike theorem requires the convergence of the myopic probabilities, which in turn commands additional assumptions. The second subsection achieves this task for a class of diffusion models with several risky assets, and with a single state variable driving investment opportunities. This class nests several models in the literature, and allows for return predictability, stochastic volatility, and stochastic interest rates.

The *explicit* turnpike – stated at the end of the second subsection – holds for the same class of diffusion models. While in the abstract and classical turnpikes the benchmark is the optimal portfolio for a isoelastic utility, but with the same finite horizon, in the explicit turnpike the benchmark is the long-run optimal portfolio, that is the optimal portfolio for asymptotic expected utility.

2.1. *The Abstract Turnpike.* Consider two investors. One of them has Constant Relative Risk Aversion (henceforth CRRA) equal to $1-p$, which corresponds to either power utility x^p/p ($0 \neq p < 1$), or to logarithmic utility $\log x$ ($p = 0$). The other investor has a generic utility function $U : \mathbb{R}_+ \to \mathbb{R}$. The marginal utility ratio $\mathfrak{R}(x)$ measures how close U is to the reference utility:

$$\mathfrak{R}(x) := \frac{U'(x)}{x^{p-1}}, \quad x > 0.$$

Assumption 2.1. The utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $U'(0) = \infty$ and $U'(\infty) = 0$. The marginal utility ratio satisfies:

$$\text{(CONV)} \qquad \lim_{x\uparrow\infty} \mathfrak{R}(x) = 1,$$

$$\text{(LB-0)} \qquad 0 < \liminf_{x\downarrow 0} \mathfrak{R}(x), \qquad \text{for } 0 \neq p < 1,$$

$$\text{(UB-0)} \qquad \limsup_{x\downarrow 0} \mathfrak{R}(x) < \infty, \qquad \text{for } p < 1.$$

Condition (CONV) entails that, when investors are rich, their marginal utilities are very close. Put differently, good outcomes have a similar impact for the two investors. Such a condition is typical of turnpike results (cf. Dybvig, Rogers, and Back (1999) and Huang and Zariphopoulou (1999)).

Conditions (LB-0) and (UB-0) require that, when investors are poor, their marginal utilities are not too far apart, in that their ratio is bounded within a positive interval. Because these conditions involve the behavior of the utility near zero, they seem less relevant for turnpike results. Yet, their importance stems from the potentially large impact of bad outcomes with small probability. Unless bad outcomes, in addition to good ones, have a comparable effect for both investors, their portfolios may remain far apart, even for long horizons.

Both investors choose from the same set $\mathcal{X}^T$ of wealth processes. Given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, \mathbb{P})$, the set $\mathcal{X}^T$ satisfies the following:

ASSUMPTION 2.2. For each $T \in \mathbb{R}_+$, $\mathcal{X}^T$ is a set of nonnegative semimartingales such that:

i) $X_0 = 1$ for all $X \in \mathcal{X}^T$;
ii) $\mathcal{X}^T$ contains some strictly positive X (i.e. $\inf_{t\in[0,T]} X_t > 0$ $\mathbb{P}$-a.s.);
iii) $\mathcal{X}^T$ is *convex*: $((1-\alpha)X + \alpha X') \in \mathcal{X}^T$ for any $X, X' \in \mathcal{X}^T$ and $\alpha \in [0,1]$;
iv) $\mathcal{X}^T$ is stable under *compounding*: if $X, X' \in \mathcal{X}^T$ with X' strictly positive and τ is a $[0,T]$-valued stopping time, then $\mathcal{X}^T$ contains the process X'' that compounds X with X' at τ:

$$X'' = X 1_{[\![0,\tau[\![} + X' \frac{X_\tau}{X'_\tau} 1_{[\![\tau,T]\!]} = \begin{cases} X_t(\omega), & \text{if } t \in [0, \tau(\omega)[\\ (X_\tau(\omega)/X'_\tau(\omega))\, X'_t(\omega), & \text{if } t \in [\tau(\omega), T] \end{cases}$$

The existence of a strictly positive wealth is a non-degeneracy condition, and is trivial in the presence of a safe asset. Convexity and compounding mean that wealth processes are stable under portfolio formation, across payoffs and over time. Using the index 0 for the CRRA investor, and 1 for the generic investor, their maximization problems are:

$$u^{0,T} = \sup_{X\in\mathcal{X}^T} \mathbb{E}^{\mathbb{P}}\left[X_T^p/p\right], \qquad u^{1,T} = \sup_{X\in\mathcal{X}^T} \mathbb{E}^{\mathbb{P}}\left[U\left(X_T\right)\right]. \tag{2.1}$$

The next assumption requires that these problems are well-posed:

ASSUMPTION 2.3. For all $T > 0$ and $i = 0, 1$, $-\infty < u^{i,T} < \infty$ and there exist an optimal $X^{i,T}$.

The central objects in the abstract turnpike theorem are the ratio of optimal wealth processes, and their stochastic logarithms:

$$r_u^T := \frac{X_u^{1,T}}{X_u^{0,T}}, \qquad \Pi_u^T := \int_0^u \frac{dr_v^T}{r_{v-}^T}, \qquad \text{for } u \in [0,T].$$

These objects are well-defined (cf. Corollary 4.2 and Remark 4.3 below). Moreover $r_0^T = 1$ since both investors have the same initial capital. Define also the *myopic probabilities* $\left\{\mathbb{P}^T\right\}_{T\geq 0}$ by:

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\left(X_T^{0,T}\right)^p}{\mathbb{E}^{\mathbb{P}}\left[\left(X_T^{0,T}\right)^p\right]}. \tag{2.2}$$

The above densities are well-defined and strictly positive (again by Remark 4.3), and $\mathbb{P}^T = \mathbb{P}$ in the logarithmic case $p = 0$. These myopic probabilities are interpreted as follows: an investor with relative risk aversion $1-p$ under the probability $\mathbb{P}$ selects the same optimal payoff as another investor under the probability $\mathbb{P}^T$, but with logarithmic utility, that is with unit risk aversion[2].

Finally, the next assumption prescribes that, as the horizon increases, payoff spaces include arbitrary large elements, in the following sense:

[2] These probabilities already appear in the work of Kramkov and Sîrbu (2006a; 2006b; 2007) under the name of $\mathbf{R}$.

ASSUMPTION 2.4. There exists a family $(\hat{X}^T)_{T\geq 0}$ such that $\hat{X}^T \in \mathcal{X}^T$ and:

(GROWTH) $$\lim_{T\to\infty} \mathbb{P}^T(\hat{X}^T_T \geq N) = 1 \qquad \text{for any } N > 0.$$

This assumption is central to turnpike results, and is readily checked in applications. It is satisfied, in particular, if the market includes a safe rate r bounded from below by a constant $\underline{r} > 0$, therefore the payoff space contains $\hat{X}^T \geq e^{\underline{r}T}$. In fact, Assumption 2.4 is satisfied even in certain markets with no safe asset, or with a zero safe rate (cf. Example 3.1 below).

With the above definitions, the abstract version of the turnpike theorem reads as follows:

THEOREM 2.5 (Abstract Turnpike). *Let Assumptions 2.1 — 2.4 hold. Then, for any $\epsilon > 0$,*

a) $\lim_{T\to\infty} \mathbb{P}^T\left(\sup_{u\in[0,T]} \left|r^T_u - 1\right| \geq \epsilon\right) = 0,$

b) $\lim_{T\to\infty} \mathbb{P}^T\left(\left[\Pi^T, \Pi^T\right]_T \geq \epsilon\right) = 0,$ *where $[\cdot,\cdot]$ denotes the square bracket of semimartingales.*

REMARK 2.6.

i) Since $\mathbb{P}^T \equiv \mathbb{P}$ for $p = 0$, convergence holds under $\mathbb{P}$ in the case of logarithmic utility. In addition, the logarithmic case does not require the condition (LB-0).

ii) For a market with asset prices $dS_u/S_u = \mu_u\, du + \sigma_u dW_u$, with $\mu_\cdot \in \mathbb{R}^d$, and a $\mathbb{R}^d$-valued Brownian motion W, the wealth processes satisfy $\frac{dX^{i,T}_u}{X^{i,T}_u} = \pi^{i,T} \frac{dS_u}{S_u}$ for $i = 0, 1$. Thus, $[\Pi^T, \Pi^T]$ measures the square distance between the portfolios $\pi^{1,T}$ and $\pi^{0,T}$:

$$\left[\Pi^T, \Pi^T\right]_\cdot = \int_0^\cdot \|\sigma\left(\pi^{1,T}_u - \pi^{0,T}_u\right)\|^2\, du.$$

iii) The theorem implies that both optimal wealth processes and portfolios are close in any time window $[0, t]$ for any fixed $t > 0$, under the probability $\mathbb{P}^T$. Indeed, for any $\epsilon, t > 0$:

$$\lim_{T\to\infty} \mathbb{P}^T\left(\sup_{u\in[0,t]} \left|r^T_u - 1\right| \geq \epsilon\right) = 0 \qquad \text{and} \qquad \lim_{T\to\infty} \mathbb{P}^T\left(\left[\Pi^T, \Pi^T\right]_t \geq \epsilon\right) = 0.$$

Except for logarithmic utility, Theorem 2.5 is not a classic turnpike theorem, in that convergence holds under the probabilities $\mathbb{P}^T$, which change with the horizon T. However, since the events $\{\sup_{u\in[0,t]} |r^T_u - 1| \geq \epsilon\}$ and $\{[\Pi^T, \Pi^T]_t \geq \epsilon\}$ are $\mathcal{F}_t$-measurable, and any such event A satisfies $\mathbb{P}^T(A) = \mathbb{E}^{\mathbb{P}}\left[1_A \left.\frac{d\mathbb{P}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t}\right]$, the relation between $\mathbb{P}^T(A)$ and $\mathbb{P}(A)$ depends on the density:

$$\left.\frac{d\mathbb{P}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{\mathbb{E}^{\mathbb{P}}_t\left[\left(X^{0,T}_T\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X^{0,T}_T\right)^p\right]}. \tag{2.3}$$

Understanding the convergence of these densities is the crucial step to bridge the gap from the abstract to the classic version of the turnpike theorem.

In fact, the densities in (2.3) become trivial under two additional assumptions: that the optimal CRRA strategy is myopic, and that its wealth process has independent returns. Under these assumptions, which hold in all the turnpike literature, with the exception of Dybvig, Rogers, and Back (1999), the density $d\mathbb{P}^T/d\mathbb{P}|_{\mathcal{F}_t}$ is independent of T, and the classic turnpike theorem follows:

COROLLARY 2.7 (IID Myopic Turnpike). *If, in addition to Assumptions 2.1 — 2.4:*

i) $X_t^{0,T} = X_t^{0,S} \equiv X_t$ *a.s. for all* $t \leq S, T$ *(myopic optimality);*
ii) X_s/X_t *and* $\mathcal{F}_t$ *are independent for all* $t \leq s$ *(independent returns).*

then, for any $\epsilon, t > 0$*:*

a) $\lim_{T\to\infty} \mathbb{P}\left(\sup_{u\in[0,t]} \left|r_u^T - 1\right| \geq \epsilon\right) = 0,$
b) $\lim_{T\to\infty} \mathbb{P}\left(\left[\Pi^T, \Pi^T\right]_t \geq \epsilon\right) = 0.$

In practice, if asset prices have independent returns, the optimal strategy for a CRRA investor entails a myopic portfolio with independent returns, and both conditions above hold. This is the case, for example, if asset prices follow exponential Lévy processes, as in Kallsen (2000). Note however, that a myopic CRRA portfolio is not sufficient to ensure that $\mathbb{P}^T$ is independent of T (cf. Example 3.2 below).

Thus, the abstract turnpike readily yields a classic turnpike theorem under the additional assumption of independent returns, which is common in the literature. However, such an additional assumption excludes the models in which portfolio choice is least tractable – and turnpike results are needed the most. This topic is discussed next in the context of diffusions.

2.2. *A Turnpike for Diffusions.* This subsection states the classic turnpike theorem for a class of diffusion models, in which a single state variable drives investment opportunities. The state variable takes values in some interval $E = (\alpha, \beta) \subseteq \mathbb{R}$, with $-\infty \leq \alpha < \beta \leq \infty$, and has the dynamics:

$$dY_t = b(Y_t)\, dt + a(Y_t)\, dW_t. \tag{2.4}$$

The market includes a safe rate $r(Y_t)$ and d risky assets, with prices S_t^i satisfying:

$$\frac{dS_t^i}{S_t^i} = r(Y_t)\, dt + dR_t^i, \quad 1 \leq i \leq d,$$

where the cumulative excess return process $R = (R^1, \cdots, R^d)'$ is defined as:

$$dR_t^i = \mu_i(Y_t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(Y_t)\, dZ_t^j, \quad 1 \leq i \leq d. \tag{2.5}$$

Here W and $Z = (Z^1, \cdots, Z^d)'$ are multivariate Wiener processes with instantaneous correlation $\rho = (\rho^1, \cdots, \rho^d)'$, i.e. $d\langle Z^i, W\rangle_t = \rho^i(Y_t)\, dt$ for $1 \leq i \leq d$, and the prime sign is for matrix transposition. Denote by $\Sigma = \sigma\sigma'$, $A = a^2$, and $\Upsilon = \sigma\rho a$. The model's coefficients are assumed regular and non-degenerate, in the following sense:

Assumption 2.8. $r \in C^{\gamma}(E, \mathbb{R})$, $b \in C^{1,\gamma}(E, \mathbb{R})$, $\mu \in C^{1,\gamma}(E, \mathbb{R}^d)$, $A \in C^{2,\gamma}(E, \mathbb{R})$, $\Sigma \in C^{2,\gamma}(E, \mathbb{R}^{d\times d})$, and $\Upsilon \in C^{2,\gamma}(E, \mathbb{R}^d)$ for some $\gamma \in (0, 1]$. For all $y \in E$, Σ is strictly positive definite and A is strictly positive.

These regularity conditions imply the local existence and uniqueness of a solution to the joint dynamics of the state variable and asset prices. To ensure the existence of a unique global solution, a further assumption is needed, along with some more notation. Let Ω be the space of continuous maps $\omega : \mathbb{R}_+ \to \mathbb{R}^{d+1}$ and $(\mathcal{B}_t)_{t\geq 0}$ be the filtration generated by the coordinate process Ξ defined by $\Xi_t(\omega) = \omega_t$ for $\omega \in \Omega$. Let $\mathcal{F} = \sigma(\Xi_t, t \geq 0)$ and $\mathcal{F}_t = \mathcal{B}_{t+}$. Setting $\xi = (z, y) \in \mathbb{R}^d \times E$, the infinitesimal generator of (R, Y) is given by:

$$L = \frac{1}{2}\sum_{i,j=1}^{d+1} \tilde{A}_{ij}(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d+1} \tilde{b}_i(\xi)\frac{\partial}{\partial\xi_i}, \quad \tilde{A} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix} \text{ and } \tilde{b} = \begin{pmatrix} \mu \\ b \end{pmatrix}. \tag{2.6}$$

DEFINITION 2.9. A family of probability measures $(\mathbb{P}^\xi)_{\xi\in\mathbb{R}^d\times E}$ on $(\Omega,\mathcal{F})$ is a solution of the martingale problem for L on $\mathbb{R}^d\times E$ if, for each $\xi\in\mathbb{R}^d\times E$: *i)* : $\mathbb{P}^\xi(\Xi_0=\xi)=1$, *ii)* : $\mathbb{P}^\xi(\Xi_t\in\mathbb{R}^d\times E,\forall t\geq 0)=1$, and *iii)* : $\left(f(\Xi_t)-f(\Xi_0)-\int_0^t Lf(\Xi_u)\,du;\mathcal{B}_t\right)$ is a $\mathbb{P}^\xi$ martingale for all $f\in C_0^2(\mathbb{R}^d\times E)$.

ASSUMPTION 2.10. The martingale problem for L is *well posed*, in that it has a unique solution.[3]

This assumption is merely technical, in that it requires that the original market is well defined. By contrast, the next assumption places some restrictions on market dynamics.

ASSUMPTION 2.11. $\rho'\rho$ is constant (i.e. it does not depend on y), and $\sup_{y\in E}c(y)<\infty$, where

$$c(y):=\frac{1}{\delta}(pr-\frac{q}{2}\mu'\Sigma^{-1}\mu)(y),\qquad y\in E,\tag{2.7}$$

with $q:=\frac{p}{p-1}$ and $\delta:=\frac{1}{1-q\rho'\rho}$.

Assumption 2.11 is straightforward to check, and holds when $p\leq 0$ for virtually all models in the literature, with the exception of correlation risk (cf. Buraschi, Porchia, and Trojani (2010)).

Recall that an *admissible portfolio* is an $(\mathcal{F}_t)_{t\geq 0}$ adapted, R-integrable process $(\pi_t)_{t\geq 0}^{1\leq i\leq d}$, representing the proportions of wealth invested in each risky asset, such that its corresponding wealth process $(X_t^\pi)_{t\geq 0}$, starting with initial capital $X_0=x>0$, satisfies the relation:

$$\frac{dX_t^\pi}{X_t^\pi}=r(Y_t)dt+\pi_t'\,dR_t.\tag{2.8}$$

The value function for the horizon $T\in\mathbb{R}_+$ is given by:

$$u^T(t,x,y)=\sup_{\pi\text{ admissible}}\mathbb{E}^{\mathbb{P}}\left[(X_T^\pi)^p/p\,|\,X_t=x,Y_t=y\right],\qquad\text{for } t\in[0,T].\tag{2.9}$$

These utility maximization problems are well posed at all horizons under the following assumption:

ASSUMPTION 2.12. There exist $(\hat v,\lambda_c)$ such that $\hat v\in C^2(E)$, $\hat v>0$, and solves the equation:

$$\mathcal{L}\,v+c\,v=\lambda\,v,\quad y\in E,\tag{2.10}$$

where $\mathcal{L}:=\frac{1}{2}A\,\partial^2_{yy}+B\,\partial_y$ and $B:=b-q\Upsilon'\Sigma^{-1}\mu$. Also, the following conditions are satisfied:

$$\int_\alpha^{y_0}\frac{1}{\hat v^2Am(y)}dy=\infty\qquad\qquad\int_{y_0}^{\beta}\frac{1}{\hat v^2Am(y)}dy=\infty,\tag{2.11}$$

$$\int_\alpha^{\beta}\hat v^2\,m(y)\,dy<\infty\qquad\qquad\int_\alpha^{\beta}\hat v\,m(y)\,dy<\infty,\tag{2.12}$$

where, for some $y_0\in E$:

$$m(y):=\frac{1}{A(y)}\exp\left(\int_{y_0}^y\frac{2B(z)}{A(z)}dz\right).\tag{2.13}$$

[3]A solution to the martingale problem for L is equivalent to a weak solution to the SDEs for (R,Y) in (2.4) and (2.5) (Rogers and Williams, 2000, Chapter V). In particular, under Assumptions 2.8 and 2.10, for any $\xi\in\mathbb{R}^d\times E$, there exists a probability measure $\mathbb{Q}^\xi$ on $(\Omega,\mathcal{F})$ and independent $(\mathbb{Q}^\xi,(\mathcal{F}_t)_{t\geq 0})$ Wiener processes B and W, with respective dimensions d and 1, such that $\big((R,Y),(B,W),(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{Q}^\xi)\big)$ is a weak solution to (2.4) and (2.5) with $(R_0,Y_0)=\xi$. Here, we have set $Z=\rho W+\bar\rho B$ where $\bar\rho$ is the positive square root of $1_d-\rho\rho'$. Because of this correspondence, solutions to martingale problems and weak solutions of the associated SDEs are used interchangeably.

This assumption is understood as follows. Equation (2.10) is the ergodic HJB equation, which controls the long-run limit of the utility maximization problem (cf. Guasoni and Robertson (2009) Theorem 7 and Section 2.2.1). Its solution $\hat{v}$ approximates the finite-horizon value functions u^T, for large T, by $u^T(x,y,0) \sim (x^p/p)(e^{\lambda T}\hat{v}(y))^\delta$ (see (5.2), Propositions 5.5 and 5.7 below for details). Thus, assuming that (2.10) has a solution guarantees that the long-run optimization problem is well posed. Note that the value of δ reflects the power transformation of Zariphopoulou (2001), which allows to write the ergodic HJB equation in the linear form (2.10).

To understand the meaning of (2.11) and (2.12), define the operator:

$$\hat{\mathcal{L}} = \frac{1}{2}\sum_{i,j=1}^{d+1} \tilde{A}_{ij}(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d+1}\hat{b}_i(\xi)\frac{\partial}{\partial\xi_i}, \quad \hat{b} = \begin{pmatrix} \frac{1}{1-p}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right) \\ B + A\frac{\hat{v}_y}{\hat{v}} \end{pmatrix}, \tag{2.14}$$

where $\tilde{A}$ is the same as in (2.6). Condition (2.11) in Assumption 2.12 implies that the martingale problem for $\hat{\mathcal{L}}$ on $\mathbb{R}^d \times E$ has a unique solution $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ (see Lemma 5.2 below), the *long-run* probability. $\hat{\mathbb{P}}^\xi$ is equivalent to $\mathbb{P}^\xi$ (Lemma 5.2, part *ii*)), and the stochastic differential equation associated to the operator $\hat{\mathcal{L}}$ has a unique weak solution starting at ξ:

$$\begin{aligned} dR_t &= \frac{1}{1-p}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right)(Y_t)\,dt + \sigma(Y_t)\,d\hat{Z}_t, \\ dY_t &= \left(B + A\frac{\hat{v}_y}{\hat{v}}\right)(Y_t)\,dt + a(Y_t)\,d\hat{W}_t, \end{aligned} \tag{2.15}$$

On the other hand, condition (2.12) implies that Y is positively recurrent under $\hat{\mathbb{P}}^\xi$ for any $\xi \in \mathbb{R}^d \times E$ (cf. Corollary 5.1.11 in Pinsky (1995)). This property is also key to understand the long-run asymptotics of $\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t}$. A simple criterion to check Assumption 2.12 is the following:

PROPOSITION 2.13. *Let Assumptions 2.8, 2.10 and 2.11 hold. If c in* (2.7) *and m in* (2.13) *satisfy:*

$$\int_\alpha^\beta m(y)dy < \infty, \tag{2.16}$$

$$\lim_{y\downarrow\alpha} c(y) = \lim_{y\uparrow\beta} c(y) = -\infty. \tag{2.17}$$

Then, Assumption 2.12 holds.

REMARK 2.14. If the interest rate r is bounded from below, and $p < 0$, (2.17) states that, the squared norm of the vector of risk premia $\sigma^{-1}\mu$ goes to ∞ at the boundary of the state space E.

A major advantage of Assumption 2.12 is to guarantee (see Proposition 5.5 below) that at all finite horizons T, the value functions u^T in (2.9) can be represented via $u^T(t,x,y) = \frac{x^p}{p}(v^T(t,y))^\delta$ for $(t,x,y) \in [0,T]\times\mathbb{R}_+\times E$, where v^T is a strictly positive classical solution to the linear parabolic PDE:

$$\begin{aligned} &\partial_t v + \mathcal{L}v + c\,v = 0, && (t,y) \in (0,T)\times E, \\ &v(T,y) = 1, && y \in E. \end{aligned} \tag{2.18}$$

Moreover, the optimal portfolio for the horizon T is (all functions are evaluated at (t, Y_t)):

$$\pi^T = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{v_y^T}{v^T}\right). \tag{2.19}$$

Thus, the wealth process corresponding to this portfolio leads to the the optimal terminal wealth $X_T^{\pi^T}$, which in turn defines the probability[4] $\mathbb{P}^{T,y}$ by (2.2). Understanding the convergence of $\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t}$ is key to go beyond the abstract version of the turnpike. To this end, observe from (2.15) that the portfolio:

$$\hat{\pi} = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right) \tag{2.20}$$

is optimal for logarithmic utility under the probability $\hat{\mathbb{P}}^y$. This fact suggests that the conditional densities of $\hat{\mathbb{P}}^y$ are natural candidates for the limits of the conditional densities $\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t}$. Combined with the positive recurrence of Y under $(\hat{\mathbb{P}}^y)_{y\in E}$ (implied by (2.12)), the next result follows:

LEMMA 2.15. *Let Assumptions 2.8, 2.10, 2.11, and 2.12 hold. Then, for all $y \in E$ and $t \geq 0$:*

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty} \frac{d\mathbb{P}^{T,y}}{d\hat{\mathbb{P}}^y}|_{\mathcal{F}_t} = 1. \tag{2.21}$$

This result essentially allows to replace $\mathbb{P}^{T,y}$ in Theorem 2.5 with probabilities $\hat{\mathbb{P}}^y$. Then the classic turnpike follows from the equivalence of $\hat{\mathbb{P}}^y$ and $\mathbb{P}^y$ (cf. Lemma 5.2, part (ii)):

THEOREM 2.16 (Classic Turnpike). *Let Assumptions 2.1 — 2.4, 2.8, and 2.10 — 2.12 hold. Then, for all $y \in E$, $0 \neq p < 1$ and $\epsilon, t > 0$:*

a) $\lim_{T\to\infty} \mathbb{P}^y\left(\sup_{u\in[0,t]} \left|r_u^T - 1\right| \geq \epsilon\right) = 0,$
b) $\lim_{T\to\infty} \mathbb{P}^y\left(\left[\Pi^T, \Pi^T\right]_t \geq \epsilon\right) = 0.$

Abstract and classic turnpikes compare the finite-horizon optimal portfolio of a generic utility to that of its CRRA benchmark at the same finite horizon. By contrast, the explicit turnpike, discussed next, uses as benchmark the long horizon limit of the optimal CRRA portfolio.

This result has two main implications: first, and most importantly, it shows that the two approximations of replacing a generic utility with a power, and a finite horizon problem with its long-run limit, lead to small errors as the horizon becomes large. Second, this theorem has a nontrivial statement even for U of CRRA type: in this case, it states that the optimal finite-horizon portfolio converges to the long-run optimal portfolio, identified as a solution to the ergodic HJB equation (2.10).

To state the explicit turnpike, define the ratio of optimal wealth processes relative to the long-run benchmark, and their stochastic logarithms as:

$$\hat{r}_u^T := \frac{X_u^{1,T}}{\hat{X}_u}, \qquad \hat{\Pi}_u^T := \int_0^u \frac{d\hat{r}_v^T}{\hat{r}_{v-}^T}, \qquad \text{for } u \in [0,T],$$

where $\hat{X}$ is the wealth process of the long-run portfolio $\hat{\pi}$. The explicit turnpike then reads as:

THEOREM 2.17 (Explicit Turnpike). *Under the assumptions of Theorem 2.16, for any $y \in E$, $\epsilon, t > 0$ and $0 \neq p < 1$:*

a) $\lim_{T\to\infty} \mathbb{P}^y\left(\sup_{u\in[0,t]} \left|\hat{r}_u^T - 1\right| \geq \epsilon\right) = 0,$
b) $\lim_{T\to\infty} \mathbb{P}^y\left(\left[\hat{\Pi}^T, \hat{\Pi}^T\right]_t \geq \epsilon\right) = 0.$

If U is of CRRA type, Assumption 2.4 is not needed for the above convergence.

[4]Since $R_0 = 0$ by assumption, $\mathbb{P}^\xi$ with $\xi = (0, y)$ is denoted as $\mathbb{P}^y$. The same convention applies to $\hat{\mathbb{P}}^\xi$.

3. Examples. The examples in this section show the relevance of the assumptions made in the statements of the main results. The first example is a simple Black-Scholes market with a zero safe rate. Its message is twofold: first, it shows that Assumption 2.4 may hold even without a positive rate. Second, it shows that replacing the myopic probabilities $\mathbb{P}^T$ with the physical probability $\mathbb{P}$ does not yield an equivalent condition, even in the simplest model.

EXAMPLE 3.1. Consider the Black-Scholes market, where the interest rate is zero and a single risky asset S satisfies $\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$ for some constants μ and $\sigma > 0$. The optimal portfolio for power utility x^p/p is $\pi = \frac{1}{1-p}\frac{\mu}{\sigma^2}$ (see Merton (1971)). Hence, the optimal wealth process satisfies $\frac{dX_t}{X_t} = \frac{1}{1-p}(\lambda^2\, dt + \lambda\, dW_t)$, where $\lambda = \mu/\sigma$ is the Sharpe ratio. Then:

$$X_T = x \exp\left(\left(1 - \frac{1}{2(1-p)}\right)\frac{\lambda^2}{1-p}T + \frac{\lambda}{1-p}W_T\right), \qquad \text{for any } T \geq 0.$$

It then follows from (2.2) that

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \mathcal{E}\left(\frac{p}{1-p}\lambda W\right)_T,$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential. Under $\mathbb{P}^T$, the optimal wealth process reads as:

$$X_T = x \exp\left(\frac{\lambda^2}{2(1-p)^2}T + \frac{\lambda}{1-p}\tilde{W}_T\right), \qquad \text{for any } T \geq 0,$$

where $\tilde{W}_t = W_t - \lambda\frac{p}{1-p}t$ is a Brownian motion under $\mathbb{P}^T$. As a result, the optimal wealth process X satisfies (2.4), as long as $\mu \neq 0$. By contrast, note that $\lim_{T\to\infty} X_T = 0$ under $\mathbb{P}$ for $p \geq 1/2$.

Next, consider a market in which returns have constant volatility, but their drift is an independent Ornstein-Uhlenbeck process. Such independence entails that optimal CRRA portfolios are myopic, as in assumption *i*) of Corollary 2.7. By contrast, the time-varying drift makes returns dependent over time, hence assumption *ii*) of the same corollary does not hold.

In this example, the conditional myopic densities $\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t}$ change with T, even though the optimal CRRA portfolio is myopic. As a result, the proof of Corollary 2.7 fails, because it requires that $\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t}$ is constant with respect to T, which in turn requires the independence of returns. Yet, both the classic and the explicit turnpikes hold in this model, in that Theorems 2.16 and 2.17 do apply. Of course, these results depend on the ergodicity of the diffusion Y.

EXAMPLE 3.2. Consider the following special case of diffusion models in Section 2.2:

$$dR_t = Y_t\, dt + dZ_t \qquad \text{and} \qquad dY_t = -Y_t\, dt + dW_t,$$

where the correlation between Z and W is $\rho = 0$ and the safe rate $r > 0$. The optimal portfolio for a CRRA investor is a myopic portfolio $\pi^T_t = \frac{Y_t}{1-p}$; see (2.19). However, the conditional density $\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t}$ depends on the horizon T. Indeed, it follows from Proposition 5.4 below that

$$\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}\left(\int \frac{v^T_y(s, Y_s)}{v^T(s, Y_s)}\, dW_s - q\int_0^\cdot Y_s\, dZ_s\right)_t,$$

where v^T satisfies the HJB equation $\partial_t v + \frac{1}{2}\partial^2_{yy}v - y\partial_y v + (rp - \frac{q}{2}y^2)v = 0$ with $v(T, y) = 1$. The above conditional density is independent of T only if $g^T(t, y) := \frac{v^T_y(t,y)}{v^T(t,y)}$ is independent of T for any fixed (t, y). It is easy to check that g^T satisfies $\partial_t g + \frac{1}{2}\partial^2_{yy} g + (g - y)\partial_y g - g - qy = 0$ with $g(T, y) = 0$. If g^T was independent of T, 0 should be a solution to the previous equation. However, this is clearly not the case for $q \neq 0$

The last example of this section shows a model in which the log optimal portfolio is optimal – at least at integer-valued horizons – for any utility function. By contrast, Huberman and Ross (1983) prove that in discrete time, and with independent returns, the classic turnpike holds if and only if the utility function satisfies a regular variation assumption. This is the only necessary and sufficient condition in the turnpike literature, and the natural question is whether a similar result holds when returns are dependent. The next example provides a negative answer, because the turnpike property holds, in a rather trivial sense, regardless of regular variation conditions.

EXAMPLE 3.3. Consider a Black-Scholes model with zero interest rate and $dS_t/S_t = \mu dt + \sigma dW_t$, with $\mu, \sigma \neq 0$. Wealth processes are defined as usual, starting from unit capital. In this model, the log-optimal wealth process X satisfies $\log X_t = (\lambda^2/2)t + \lambda W_t$ for $t \in \mathbb{R}_+$, where $\lambda = \mu/\sigma$. Define a sequence of random horizons $(\tau_n)_{n\in\mathbb{N}}$ setting $\tau_n = \inf\{t \in \mathbb{R}_+ \mid X_t = n\}$. It is clear that $\mathbb{P}(\tau_n < \infty) = 1$ for all $n \in \mathbb{N}$, and that $\mathbb{P}(\lim_{n\to\infty} \tau_n = \infty) = 1$.

By the numéraire property of the log-optimal portfolio, any wealth process ξ with $\xi_0 = 1$ satisfies $\mathbb{E}^{\mathbb{P}}[\xi_{\tau_n}/X_{\tau_n}] \leq 1$ or, equivalently, $\mathbb{E}^{\mathbb{P}}[\xi_{\tau_n}] \leq n$. Therefore, for an investor with *any* concave and increasing utility function U, Jensen's inequality gives $\mathbb{E}^{\mathbb{P}}[U(\xi_{\tau_n})] \leq U(\mathbb{E}^{\mathbb{P}}[\xi_{\tau_n}]) \leq U(n) = \mathbb{E}^{\mathbb{P}}[U(X_{\tau_n})]$. In other words, for every time-horizon τ_n, $n \in \mathbb{N}$, every investor's optimal wealth is the same.

The previous construction can be adapted to become a sequence of deterministic times via a time-change. To wit, set $L = \log X$ and consider the strictly increasing, adapted process A given by:

$$A_t = n - \exp\left(-\int_{\tau_{n-1}}^{t} \frac{1}{(n - L_u)^2} du\right), \text{ for } t \in [\tau_{n-1}, \tau_n).$$

We claim that, $\mathbb{P}$-a.s, $A_{\tau_n} = n$ holds for all $n \in \mathbb{N}$. Indeed, for all $n \in \mathbb{N}$, the process χ^n defined via $\chi^n_t = n - L_{\tau_n \wedge t}$ for $t \in \mathbb{R}_+$ is a Brownian motion with $\chi^n_0 = 1$, drift rate $-\lambda^2/2$ and volatility λ, stopped when it reaches zero, the latter happening at time τ_n. Since

$$\log(\chi^n_t) = \int_{\tau_{n-1}}^{t} \frac{1}{\chi^n_u} d\chi^n_u - \frac{\lambda^2}{2}\int_{\tau_{n-1}}^{t} \frac{1}{(\chi^n_u)^2} du \text{ holds for } t \in [\tau_{n-1}, \tau_n)$$

(note that $\chi^n_{\tau_{n-1}} = 1$), on the event $\left\{\int_{\tau_{n-1}}^{\tau_n} \frac{1}{(n-L_u)^2} du < \infty\right\} = \left\{\int_{\tau_{n-1}}^{\tau_n} \frac{1}{(\chi^n_u)^2} du < \infty\right\}$ one would have $\chi^n_{\tau_n} > 0$; in view of $\mathbb{P}\left[\chi^n_{\tau_n} = 0\right] = 1$, $\mathbb{P}\left[A_{\tau_n} = n\right] = 1$ for all $n \in \mathbb{N}$ has to hold. Continuing, define C as the inverse of A, that is $C_t = \inf\{u \in \mathbb{R}_+ \mid A_u \geq t\}$ for $t \in \mathbb{R}_+$. Note that C is strictly increasing, $C_n = \tau_n$ for all $n \in \mathbb{N}$, and each C_t is a stopping time. Consider now a new price process $\widetilde{S}$ adapted to the filtration $\widetilde{\mathbf{F}} = (\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$, where $\widetilde{S}_t = S_{C_t}$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_{C_t}$ for all $t \in \mathbb{R}_+$. In this setting, the log-optimal portfolio $\widetilde{X}$ satisfies $\widetilde{X}_t = X_{C_t}$ for $t \in \mathbb{R}_+$; in particular, $\widetilde{X}_n = X_{C_n} = X_{\tau_n} = n$ for all $n \in \mathbb{N}$. Again, for every time-horizon $n \in \mathbb{N}$, the log optimal wealth process remains optimal for any investor with an integer horizon n.

4. Proof of the Abstract Turnpike. This section contains the results leading to the abstract version of the turnpike theorem. The proof proceeds through three main steps:

i) Derive the first order condition for optimal payoffs;
ii) Show that optimal payoffs for the generic utility converge to their CRRA counterparts;

iii) Obtain from the convergence of optimal payoffs the convergence of wealth processes.

The first subsection studies the first order condition and its implications. The second subsection derives the convergence of optimal payoffs, and is divided into two parts. The first part treats logarithmic utility, which is technically simpler, and leads to a slightly stronger result. The second part deals with power utility. These two subsections make no reference to market structure, and can be thought of as results on sequences of one-period models. By contrast, the identification of payoffs as terminal values of wealth processes is the main topic in the third subsection.

4.1. *The First Order Condition.* The first order condition is commonly used in the literature as an assumption, or as an ansatz to find a candidate optimal payoff. By contrast, the first order condition in this paper is derived under well-posedness of the finite horizon problem, and under the assumption that the utility function is close to a member of the CRRA class.

Since the marginal utility ratio $\mathfrak{R}$ is continuous on $\mathbb{R}_+$, (UB-0) and (CONV) imply that:

$$\text{(UB)} \qquad \overline{\mathfrak{R}} := \sup_{x>0} \mathfrak{R}(x) < \infty;$$

Similarly, (LB-0) and (CONV) imply that:

$$\text{(LB)} \qquad \underline{\mathfrak{R}} := \inf_{x>0} \mathfrak{R}(x) > 0.$$

Denote by $\mathcal{C}^T = \{X_T : X \in \mathcal{X}^T\}$ the set of final payoffs of the class of wealth processes $\mathcal{X}^T$. Note that $\mathcal{C}^T \subseteq \mathbb{L}^0_+$ is convex, and, setting $\mathbb{L}^0_{++} = \{f \in L^0_T | \mathbb{P}(f > 0) = 1\}$, $\mathcal{C}^T \cap \mathbb{L}^0_{++} \neq \emptyset$ for all $T \geq 0$. The first order condition reads as follows:

LEMMA 4.1. *Let* (UB) *hold and, if $p \neq 0$ let also* (LB) *hold. If Assumption 2.3 holds, then $\mathbb{E}^{\mathbb{P}}[(f^{1,T})^p] < \infty$, where $f^{1,T} := X^{1,T}_T$, and:*

$$\mathbb{E}^{\mathbb{P}}\left[U'(f^{1,T})(f)\right] \leq \mathbb{E}^{\mathbb{P}}\left[U'(f^{1,T})(f^{1,T})\right] < \infty, \qquad \textit{for any } f \in \mathcal{C}^T. \tag{4.1}$$

PROOF. In this proof, all expectations are under $\mathbb{P}$ and the superscript 1 is omitted. For any $f \in \mathcal{C}^T$, fix $\epsilon \in (0, 1/2)$. By the convexity of $\mathcal{C}^T$, $f^T + \epsilon(f - f^T) \in \mathcal{C}^T$. The optimality of f^T implies that:

$$\mathbb{E}\left[\frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right]\right] \leq 0.$$

There exists an integrable random variable g such that $\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right]/\epsilon \geq g$ for any $\epsilon \in (0, 1/2)$. Indeed, the concavity of U implies that:

$$\begin{aligned}\frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right] &\geq U'(f^T + \epsilon(f - f^T))(f - f^T)1_{\{f<f^T\}} \\ &= \mathfrak{R}(f^T + \epsilon(f - f^T))\frac{f - f^T}{(f^T + \epsilon(f - f^T))^{1-p}}1_{\{f<f^T\}},\end{aligned} \tag{4.2}$$

where $f^T + \epsilon(f - f^T) > 0$ when $0 \leq f < f^T$. Now, note that:

$$\frac{f - f^T}{(f^T + \epsilon(f - f^T))^{1-p}} \geq -(1-\epsilon)^{p-1}(f^T)^p \qquad \text{for any } f \text{ such that } 0 \leq f < f^T. \tag{4.3}$$

Indeed, define $F(x) := \frac{x}{(f^T + \epsilon x)^{1-p}}$ for $x \in [-f^T, 0)$. Next, estimate the numerator of its derivative $F'(x) = \frac{f^T + p\epsilon x}{(f^T + \epsilon x)^{2-p}}$. For $p \leq 0$, the numerator is clearly nonnegative for $x \in [-f^T, 0)$. For $0 < p < 1$,

$f^T + p\epsilon x \geq f^T - p\epsilon f^T \geq 0$ because $1 - p\epsilon > 0$. Therefore, as an increasing function on $[-f^T, 0)$, F attains its minimum at $-f^T$ on $[-f^T, 0)$. Hence the claim follows. Now, combining (4.2), (4.3), and $\epsilon \in (0, 1/2)$, it follows that:

$$\frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right] \geq -2^{1-p}\,\overline{\mathfrak{R}}\,(f^T)^p =: g.$$

The above inequality also holds when $f^T = 0$. In the next paragraph, we shall show that g is integrable for different values of p.

It suffices to show that $(f^T)^p$ is integrable. This is trivial for $p = 0$. For $0 < p < 1$, (LB) implies that $U'(x) \geq \underline{\mathfrak{R}}\, x^{p-1}$ for $x \in \mathbb{R}_{++}$. Integrating this inequality on (x_*, x) for some fixed constant x_* gives

$$\frac{1}{\underline{\mathfrak{R}}}\left(U(x) - U(x_*)\right) \geq \frac{1}{p}(x^p - x_*^p).$$

Therefore on the set $\{f^T > x_*\}$,

$$(f^T)^p \leq x_*^p + \frac{p}{\underline{\mathfrak{R}}}\left(U(f^T) - U(x_*)\right).$$

Taking expectation on both sides, and recalling with the assumption $\mathbb{E}[U(f^T)] < \infty$, it follows that $\mathbb{E}[(f^T)^p 1_{\{f^T > x_*\}}] < \infty$. On the other hand, $\mathbb{E}[(f^T)^p 1_{\{f^T \leq x_*\}}] \leq x_*^p$ for $0 < p < 1$. Hence the claim $\mathbb{E}[(f^T)^p] < \infty$ follows. For $p < 0$, a similar argument applies, while integrating (LB) on (x, x_*).

Now (UB) and $\mathbb{E}[(f^T)^p] < \infty$ combined implies that

$$\mathbb{E}\left[U'(f^T)f^T\right] \leq \overline{\mathfrak{R}}\mathbb{E}\left[(f^T)^p\right] < \infty. \tag{4.4}$$

On the other hand, the lower bound g yields:

$$\begin{aligned}
\liminf_{\epsilon \downarrow 0} \mathbb{E}\left[\frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right] - g\right] &= \liminf_{\epsilon \downarrow 0} \mathbb{E}\left[\frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right] - g\right] \\
&\geq \mathbb{E}\left[\liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon}\left[U(f^T + \epsilon(f - f^T)) - U(f^T)\right] - g\right] \\
&= \mathbb{E}\left[U'(f^T)(f - f^T) - g\right],
\end{aligned}$$

in which the inequality follows from Fatou's lemma. Then adding $\mathbb{E}[g]$ on both sides of previous inequality, noting that $-\infty < \mathbb{E}[g] < \infty$, it follows that $\mathbb{E}\left[U'(f^T)(f - f^T)\right] \leq 0$. Therefore (4.4) and the previous inequality combined implies (4.1). $\square$

As a consequence of the first order condition, the optimizer $f^{1,T}$ is strictly positive $\mathbb{P}$-a.s.

Corollary 4.2. *Under the assumptions of Lemma 4.1, $f^{1,T} > 0$, $\mathbb{P}$-a.s..*

Proof. By contradiction, suppose that $\mathbb{P}(f^{1,T} = 0) > 0$. Then, for any $f \in \mathbb{L}^0_{++} \cap \mathcal{C}^T$, it would follow that:

$$\infty = \mathbb{E}^{\mathbb{P}}\left[U'(f^{1,T})f\right] \leq \mathbb{E}^{\mathbb{P}}\left[U'(f^{1,T})f^{1,T}\right] < \infty,$$

in which the first equality follows from $U'(0) = \infty$, the rest inequalities hold thanks to Lemma 4.1. $\square$

Remark 4.3. Using the same reasoning, Lemma 4.1 and Corollary 4.2 also hold for CRRA utility x^p/p and its optimal payoff $f^{0,T}$.

4.2. *Logarithmic Utility.* This section derives the convergence of terminal payoffs, in the case of a reference investor with logarithmic utility, corresponding to $p = 0$. Then, $\mathfrak{R}$ is the ratio between the marginal utility of U and that of logarithmic utility. Recall that $\mathbb{P}^T \equiv \mathbb{P}$ for $p = 0$, (2.4) is equivalent to the following assumption:

(GROWTH-L) There exists a sequence $(h^T)_{T\geq 0}$, with $h^T \in \mathcal{C}^T$, such that $\mathbb{P}\text{-}\lim_{T\to\infty} h^T = \infty$.

The existence of such an unbounded sequence of payoffs implies that the sequence of optimal payoffs $(f^{i,T})_{T\geq 0}$ is also unbounded:

LEMMA 4.4. *Let* (UB) *and* (GROWTH-L) *hold. Then* $\mathbb{P}\text{-}\lim_{T\to\infty} f^{i,T} = \infty$ *for* $i = 0, 1$.

PROOF. Only $i = 1$ case is proved here, $i = 0$ case follows similarly. Choosing f in (4.1) as h^T in (GROWTH-L), it follows that $\mathbb{E}^{\mathbb{P}}[U'(f^{1,T})h^T] \leq \mathbb{E}^{\mathbb{P}}[U'(f^{1,T})f^{1,T}]$. Here the right-hand-side of the last inequality is bounded from above by $\overline{\mathfrak{R}}$ thanks to (UB). As a result, $\mathbb{P}\text{-}\lim_{T\to\infty} U'(f^{1,T}) = 0$, if $\mathbb{P}\text{-}\lim_{T\to\infty} h^T = \infty$. Since $U'(x) > 0$ for $x \in \mathbb{R}_{++}$ and $\lim_{x\uparrow\infty} U'(x) = 0$, it follows that $\mathbb{P}\text{-}\lim_{T\to\infty} f^{1,T} = \infty$. □

Even though both $f^{0,T}$ and $f^{1,T}$ are unbounded as $T \to \infty$, the following main result of this subsection shows that their ratio $r^T := r^T_T = \frac{f^{1,T}}{f^{0,T}}$ converges.

THEOREM 4.5. *Assume that* (UB-0), (CONV), *and* (GROWTH-L) *are satisfied. Then*

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}\left[|r^T - 1|\right] = 0. \tag{4.5}$$

The proof of this result depends on the following lemma.

LEMMA 4.6. *Let* (UB) *hold. Then*

$$\mathbb{E}^{\mathbb{P}}\left[\left|(r^T - 1)\left(1 - \frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right|\right] \leq 2\,\mathbb{E}^{\mathbb{P}}\left[\left(\mathfrak{R}(f^{1,T}) - 1\right)^2\right]. \tag{4.6}$$

In addition, if (UB-0), (CONV), *and* (GROWTH-L) *are satisfied, then*

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}\left[\left|(r^T - 1)\left(1 - \frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right|\right] = 0. \tag{4.7}$$

PROOF. All expectations are taken under $\mathbb{P}$ in this proof. Applying the first order condition to $\log x$ and U respectively[5], it follows that:

$$\mathbb{E}\left[r^T - 1\right] \leq 0 \quad \text{and} \quad \mathbb{E}\left[\frac{\mathfrak{R}(f^{1,T})}{r^T}\left(1 - r^T\right)\right] \leq 0, \quad \text{for any } T \geq 0.$$

The first inequality is known as the *numéraire* property of $f^{0,T}$. The sum of these inequalities yields

$$\mathbb{E}\left[(r^T - 1)\left(1 - \frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right] \leq 0. \tag{4.8}$$

[5](UB) is required for the first order condition.

Observe that $\left(r^T-1\right)\left(1-\mathfrak{R}(f^{1,T})/r^T\right)$ is negative, if and only if $1<r^T<\mathfrak{R}(f^{1,T})$ or $\mathfrak{R}(f^{1,T})<r^T<1$. In either cases, setting $x_-=\max(0,-x)$ and $x_+=\max(0,x)$:

$$\left(\left(r^T-1\right)\left(1-\frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right)_-\le\left(\mathfrak{R}(f^{1,T})-1\right)^2. \tag{4.9}$$

On the other hand, it follows from (4.8) that

$$\mathbb{E}\left[\left(\left(r^T-1\right)\left(1-\frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right)_+\right]\le\mathbb{E}\left[\left(\left(r^T-1\right)\left(1-\frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right)_-\right]\le\mathbb{E}\left[\left(\mathfrak{R}(f^{1,T})-1\right)^2\right]. \tag{4.10}$$

As a result, the first statement follows from (4.10).

Combining (CONV) and Lemma 4.4, we have $\mathbb{P}\text{-}\lim_{T\to\infty}\mathfrak{R}(f^{1,T})=1$. Note that $\mathfrak{R}(f^{1,T})$ is bounded from above thanks to (UB). The second statement follows from applying the bounded convergence theorem to the right-hand-side of (4.6). □

Proof of Theorem 4.5. All expectations are taken under $\mathbb{P}$ in this proof. It suffices to prove

$$\mathbb{P}\text{-}\lim_{T\to\infty}r^T=1. \tag{4.11}$$

Indeed, note that $r^T-1\ge-1$, Fatou's lemma implies that

$$0=\mathbb{E}[\liminf_{T\to\infty}r^T-1]\le\liminf_{T\to\infty}\mathbb{E}[r^T-1]\le0,$$

in which the first equality follows from (4.11) and the second inequality holds by the numéraire property of $f^{0,T}$. As a result of the last sequence of inequalities, $\lim_{T\to\infty}\mathbb{E}[r^T]=1$, which leads to (4.5) after appealing to Scheffé's lemma.

To prove (4.11), observe that (CONV) and Lemma 4.4 imply

$$\lim_{T\to\infty}\mathbb{P}\left(\left|\mathfrak{R}(f^{1,T})-1\right|\ge\delta\right)=0,\qquad\text{for any }\delta>0. \tag{4.12}$$

On the other hand, for fixed $\epsilon>0$ and $T\ge0$, consider the set

$$D^T=\left\{|r^T-1|>\epsilon,(1-\epsilon)^{1/2}\le\mathfrak{R}(f^{1,T})\le(1+\epsilon)^{1/2}\right\}.$$

Next, derive a lower bound for $\left|1-\mathfrak{R}(f^{1,T})/r^T\right|$ on D^T. If $r^T>1+\epsilon$, then $\mathfrak{R}(f^{1,T})/r^T\le(1+\epsilon)^{-1/2}<1$, hence $1-\mathfrak{R}(f^{1,T})/r^T\ge1-(1+\epsilon)^{-1/2}>0$. If $r^T<1-\epsilon$, then $\mathfrak{R}(f^{1,T})/r^T\ge(1-\epsilon)^{-1/2}>1$, hence $1-\mathfrak{R}(f^{1,T})/r^T\le1-(1-\epsilon)^{-1/2}<0$. Denoting $\eta=\min\{1-(1+\epsilon)^{-1/2},(1-\epsilon)^{-1/2}-1\}>0$, from the previous two cases we conclude that

$$\left|1-\frac{\mathfrak{R}(f^{1,T})}{r^T}\right|\ge\eta\quad\text{ on }D^T.$$

This implies

$$\mathbb{E}\left[\left|(r^T-1)\left(1-\frac{\mathfrak{R}(f^{1,T})}{r^T}\right)\right|\right]\ge\epsilon\,\eta\,\mathbb{P}(D^T).$$

Combining previous estimate with (4.7), it follows that $\lim_{T\to\infty}\mathbb{P}(D^T)=0$. Now note that

$$\mathbb{P}(|r^T-1|>\epsilon)\le\mathbb{P}(D^T)+\mathbb{P}(|\mathfrak{R}(f^{1,T})-1|\ge\delta),$$

where $\delta:=\min\{(1+\epsilon)^{1/2}-1,1-(1-\epsilon)^{1/2}\}$. Sending $T\to\infty$, it follows that $\lim_{T\to\infty}\mathbb{P}(|r^T-1|>\epsilon)=0$ from (4.12) and $\lim_{T\to\infty}\mathbb{P}(D^T)=0$. □

4.3. *Power utility case.* This subsection presents a version of Theorem 4.5 for power utility. Here $\mathfrak{R}$ is the ratio between marginal utilities of U and x^p/p with $0 \neq p < 1$.

Recall the probability measure $\mathbb{P}^T$ defined in (2.2). The optimizer $f^{0,T}$ has the numéraire property under $\mathbb{P}^T$, i.e., $\mathbb{E}^{\mathbb{P}^T}\left[f/f^{0,T}\right] \leq 1$ for any $f \in \mathcal{C}^T$. This claim follows by the inequality $\mathbb{E}^{\mathbb{P}}\left[(f^{0,T})^p\left(\frac{f}{f^{0,T}} - 1\right)\right] \leq 0$, obtained by applying the first order condition to the utility x^p/p, and switching the expectation from $\mathbb{P}$ to $\mathbb{P}^T$.

Thus, the power utility case is linked to the logarithmic case under the sequence of measures $\left\{\mathbb{P}^T\right\}_{T\geq 0}$. In this setting, Assumption 2.4 takes the following form:

(GROWTH-P)

$$\text{There exists } \left\{h^T\right\}_{T\geq 0} \text{ such that } h^T \in \mathcal{C}^T \text{ and } \lim_{T\to\infty} \mathbb{P}^T(h^T \geq N) = 1 \text{ for any } N > 0.$$

The analogue of Theorem 4.5 for power utility reads as follows:

THEOREM 4.7. *Assume that* (UB-0), (LB-0), (CONV)*, and* (GROWTH-P) *are satisfied. Then*

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r^T - 1|\right] = 0.$$

As in the previous subsection, the proof of this theorem requires some similar preliminary lemmas. The following one is the analogue of Lemma 4.4.

LEMMA 4.8. *Let* (UB), (LB)*, and* (GROWTH-P) *hold. Then*

$$\lim_{T\to\infty} \mathbb{P}^T\left(f^{0,T} \geq N\right) = 1, \quad \textit{for any } N > 0. \tag{4.13}$$

PROOF. It suffices to prove $\limsup_{T\to\infty} \mathbb{P}^T(f^{0,T} < N) = 0$ for each fixed N. To this end, the numéraire property implies that:

$$1 \geq \mathbb{E}^{\mathbb{P}^T}\left[\frac{h^T}{f^{0,T}}\right] \geq \mathbb{E}^{\mathbb{P}^T}\left[\frac{h^T}{f^{0,T}} 1_{\{f^{0,T}<N, h^T\geq\tilde{N}\}}\right] \geq \frac{\tilde{N}}{N} \mathbb{P}^T\left(f^{0,T} < N, h^T \geq \tilde{N}\right),$$

for any positive constant $\tilde{N}$. As a result, $\mathbb{P}^T(f^{0,T} < N, h^T \geq \tilde{N}) \leq N/\tilde{N}$. Combining the last inequality with (GROWTH-P), we have

$$\limsup_{T\to\infty} \mathbb{P}^T(f^{0,T} < N) \leq \limsup_{T\to\infty} \mathbb{P}^T(f^{0,T} < N, h^T \geq \tilde{N}) + \lim_{T\to\infty} \mathbb{P}^T(h^T < \tilde{N}) \leq \frac{N}{\tilde{N}}.$$

Then, the statement follows since $\tilde{N}$ is chosen arbitrarily. $\square$

At this point, introducing the analogue of Lemma 4.6 requires some notation. Define:

$$I(y) := (U')^{-1}(y) \quad \text{for } y > 0 \quad \text{and} \quad H(x) := I(x^{p-1}) \quad \text{for } x > 0.$$

LEMMA 4.9. *Let* (UB-0) *and* (CONV) *hold. Then*

$$\lim_{x\uparrow\infty} \frac{H(x)}{x} = 1 \quad \textit{and} \quad \mathfrak{H} := \sup_{x>0}\left|\frac{H(x)}{x} - 1\right| < \infty. \tag{4.14}$$

PROOF. Note that $H(x)/x$ is positive and continuous on $\mathbb{R}_{++}$. The second statement follows from the claim $\limsup_{x\uparrow\infty} H(x)/x < \infty$ and $\limsup_{x\downarrow 0} H(x)/x < \infty$, which is proved in the next two paragraphs.

Denote $y = x^{p-1}$, and observe that:

$$\frac{H(x)}{x} = \frac{I(y)}{y^{\frac{1}{p-1}}} = \left(\frac{I(y)^{p-1}}{U'(I(y))}\right)^{\frac{1}{p-1}}.$$

If $x \uparrow \infty$, due to $p < 1$ and the Inada conditions, we have $y \downarrow 0$, hence $I(y) \uparrow \infty$. As a result,

$$\lim_{x\uparrow\infty} \frac{H(x)}{x} = \lim_{y\downarrow 0} \left(\frac{I(y)^{p-1}}{U'(I(y))}\right)^{\frac{1}{p-1}} = 1,$$

in which the second equality follows from (CONV).

When x is close to zero, denote by $z = U'(x)$. Then (UB) implies that $z\,(I(z))^{1-p} \le \overline{\mathfrak{R}}$, whence $I(z) \le \left(\frac{\overline{\mathfrak{R}}}{z}\right)^{\frac{1}{1-p}}$ since $p < 1$. As a result, $A := \limsup_{z\uparrow\infty} \frac{I(z)}{z^{\frac{1}{p-1}}} \le \overline{\mathfrak{R}}^{\frac{1}{1-p}}$, which yields

$$\limsup_{x\downarrow 0} \frac{H(x)}{x} = \limsup_{y\uparrow\infty} \frac{I(y)}{y^{\frac{1}{p-1}}} = A < \infty.$$

This concludes the proof. □

This is the analogue of Lemma 4.6.

LEMMA 4.10. *Assume that* (UB) *and* (LB) *are satisfied. Then*

$$\mathbb{E}^{\mathbb{P}^T}\left[\left|\left(1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right)\right| |r^T - 1|\right] \le 2\,\mathbb{E}^{\mathbb{P}^T}\left[\left(1 - \mathfrak{R}(f^{0,T})\right)\left(1 - \frac{H(f^{0,T})}{f^{0,T}}\right)\right]. \tag{4.15}$$

Moreover, if (UB-0), (LB-0), (CONV), *and* (GROWTH-P) *are satisfied, then*

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[\left|\left(1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right)\right| |r^T - 1|\right] = 0. \tag{4.16}$$

PROOF. Applying the first order condition to utilities x^p/p and U respectively, and summing the resulting inequalities, it follows that $\mathbb{E}^{\mathbb{P}}\left[(f^{0,T})^p\left(1 - \frac{U'(f^{1,T})}{(f^{0,T})^{p-1}}\right)\left(r^T - 1\right)\right] \le 0$. After changing to the measure $\mathbb{P}^T$, the previous inequality reads

$$\mathbb{E}^{\mathbb{P}^T}\left[\left(1 - \frac{U'(f^{1,T})}{(f^{0,T})^{p-1}}\right)\left(r^T - 1\right)\right] \le 0.$$

Observe that $\left(1 - \frac{U'(f^{1,T})}{(f^{0,T})^{p-1}}\right)\left(r^T - 1\right) \le 0$ if and only if $f^{0,T} \le f^{1,T} \le H(f^{0,T})$ or $H(f^{0,T}) \le f^{1,T} \le f^{0,T}$. Thus, the following estimate, similar to (4.9), follows:

$$\left[\left(1 - \frac{U'(f^{1,T})}{(f^{0,T})^{p-1}}\right)\left(r^T - 1\right)\right]_- \le \left(1 - \mathfrak{R}(f^{0,T})\right)\left(1 - \frac{H(f^{0,T})}{f^{0,T}}\right).$$

Note that the right-hand-side of this inequality is nonnegative because $U'(f^{0,T}) \le (f^{0,T})^{p-1}$ if and only if $f^{0,T} \ge H(f^{0,T})$. Now (4.15) follows from the same argument after (4.9).

The second statement follows once we prove that

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[\left(1-\mathfrak{R}(f^{0,T})\right)\left(1-\frac{H(f^{0,T})}{f^{0,T}}\right)\right] = 0.$$

In view of (4.14), for any $\epsilon > 0$, there exists a sufficiently large N_ϵ, such that $\left|1-\frac{H(x)}{x}\right| \le \epsilon$ for any $x \ge N_\epsilon$. Then we estimate

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\left[\left(1-\mathfrak{R}(f^{0,T})\right)\left(1-\frac{H(f^{0,T})}{f^{0,T}}\right)\right]\\
&\le \mathbb{E}^{\mathbb{P}^T}\left[\left|1-\mathfrak{R}(f^{0,T})\right|\left|1-\frac{H(f^{0,T})}{f^{0,T}}\right| 1_{\{f^{0,T}\ge N_\epsilon\}}\right] + \mathfrak{H}(1+\overline{\mathfrak{R}})\, \mathbb{P}^T(f^{0,T} < N_\epsilon)\\
&\le \epsilon\,(1+\overline{\mathfrak{R}}) + \mathfrak{H}(1+\overline{\mathfrak{R}})\, \mathbb{P}^T(f^{0,T} < N_\epsilon),
\end{aligned}$$

in which the first inequality holds due to (4.14). Now, sending $T \to \infty$,

$$\limsup_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[\left(1-\mathfrak{R}(f^{0,T})\right)\left(1-\frac{H(f^{0,T})}{f^{0,T}}\right)\right] \le \epsilon(1+\overline{\mathfrak{R}})+\mathfrak{H}(1+\overline{\mathfrak{R}}) \limsup_{T\to\infty} \mathbb{P}^T(f^{0,T} < N_\epsilon) = \epsilon(1+\overline{\mathfrak{R}}),$$

in which the last equality follows from (4.13). Since the choice of ϵ is arbitrary, the claim then follows. □

As a corollary of the above result, r^T is bounded away from both 0 and ∞ in the limit $T \to \infty$.

COROLLARY 4.11. *Let* (UB-0), (LB-0), (CONV), *and* (GROWTH-P) *hold. Then there exists a sufficiently large N such that*

$$\lim_{T\to\infty} \mathbb{P}^T\left(\frac{1}{N} \le r^T \le N\right) = 1. \tag{4.17}$$

PROOF. Choose $N > \overline{\mathfrak{R}}^{\frac{1}{1-p}} \vee \underline{\mathfrak{R}}^{\frac{1}{p-1}} \vee 1$. We will show in this paragraph that $\lim_{T\to\infty} \mathbb{P}^T(r^T > N) = 0$. Since $N > 1$ and $\overline{\mathfrak{R}}N^{p-1} < 1$ (note $p < 1$), then $\frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}} < \overline{\mathfrak{R}}N^{p-1} < 1$ on the set $\{r^T > N\}$. It then follows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^T}\left[\left|1-\frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| |r^T-1|\right] &\ge \mathbb{E}^{\mathbb{P}^T}\left[\left|1-\frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| |r^T-1| 1_{\{r^T>N\}}\right]\\
&\ge (1-\overline{\mathfrak{R}}N^{p-1})(N-1)\, \mathbb{P}^T(r^T > N).
\end{aligned}$$

Combining the last estimate with (4.16), it follows that $\lim_{T\to\infty} \mathbb{P}^T(r^T > N) = 0$.

To prove $\lim_{T\to\infty} \mathbb{P}^T(r^T < 1/N) = 0$, observe that on $\{r^T < 1/N\}$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^T}\left[\left|1-\frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| |r^T-1|\right] &\ge \mathbb{E}^{\mathbb{P}^T}\left[\left|1-\frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| |r^T-1|\, 1_{\{r^T<1/N\}}\right]\\
&\ge (\underline{\mathfrak{R}}N^{1-p}-1)\left(1-\frac{1}{N}\right) \mathbb{P}^T(r^T < 1/N),
\end{aligned}$$

in which the second inequality follows from (LB) and $\underline{\mathfrak{R}}N^{1-p} > 1$. Combining the last estimate with with (4.16), it follows that $\lim_{T\to\infty} \mathbb{P}^T(r^T < 1/N) = 0$. This concludes the proof. □

Combining Lemma 4.8 and Corollary 4.11, it follows that:

$$\lim_{T\to\infty} \mathbb{P}^T(f^{1,T} \geq N) = 1, \quad \text{for sufficiently large } N. \tag{4.18}$$

Indeed, observe that $\mathbb{P}^T(f^{1,T} \geq N) \geq \mathbb{P}^T(r^T \geq 1/N; f^{0,T} \geq N^2)$, then (4.18) follows from (4.13) and (4.17).

In what follows, we will show that r^T converges to 1 under $\mathbb{P}^T$ as $T \to \infty$.

PROPOSITION 4.12. *Let* (UB-0), (LB-0), (CONV), *and* (GROWTH-P) *hold. Then*

$$\lim_{T\to\infty} \mathbb{P}^T(|r^T - 1| \geq \epsilon) = 0, \quad \textit{for all } \epsilon > 0. \tag{4.19}$$

PROOF. First note that, from (CONV) and (4.18):

$$\lim_{T\to\infty} \mathbb{P}^T(|\mathfrak{R}(f^{1,T}) - 1| \geq \delta) = 0, \text{ for any } \delta > 0. \tag{4.20}$$

Indeed, for any $\delta > 0$, due to (CONV), there exists a sufficiently large N_δ such that $|\mathfrak{R}(x) - 1| < \delta$ for any $x > N_\delta$. As a result, $\mathbb{P}^T(|\mathfrak{R}(f^{1,T}) - 1| \geq \delta, f^{1,T} > N_\delta) = 0$. Combining this with $\lim_{T\to\infty} \mathbb{P}^T(f^{1,T} \leq N_\delta) = 0$ from (4.18), we confirm the claim.

Now take $\epsilon \in (0, 1)$ and consider the set

$$D^T = \left\{|r^T - 1| \geq \epsilon, (1-\epsilon)^{\frac{1-p}{2}} \leq \mathfrak{R}(f^{1,T}) \leq (1+\epsilon)^{\frac{1-p}{2}}\right\}.$$

In the following, we will estimate the lower bound of $\left|1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right|$ on D^T for the cases $r^T > 1 + \epsilon$ and $r^T < 1 - \epsilon$ separately.

For $r^T > 1 + \epsilon$, we have $\mathfrak{R}(f^{1,T})(r^T)^{p-1} < (1+\epsilon)^{\frac{p-1}{2}} < 1$ on D^T whence

$$1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}} \geq 1 - (1+\epsilon)^{\frac{p-1}{2}} > 0;$$

For $r^T < 1 - \epsilon$, we have $\mathfrak{R}(f^{1,T})(r^T)^{p-1} > (1-\epsilon)^{\frac{p-1}{2}} > 1$ on D^T whence

$$1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}} \leq 1 - (1-\epsilon)^{\frac{p-1}{2}} < 0.$$

Denote $\eta = \min\left\{1 - (1+\epsilon)^{\frac{p-1}{2}}, -1 + (1-\epsilon)^{\frac{p-1}{2}}\right\}$. In either of the above cases, $\left|1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| \geq \eta$, therefore

$$\mathbb{E}^{\mathbb{P}^T}\left[\left|1 - \frac{\mathfrak{R}(f^{1,T})}{(r^T)^{1-p}}\right| |r^T - 1|\right] \geq \epsilon\, \eta\, \mathbb{P}^T(D^T).$$

Combining the previous inequality with (4.16), it follows that

$$\lim_{T\to\infty} \mathbb{P}^T(D^T) = 0. \tag{4.21}$$

Now, combining (4.17), (4.20) and (4.21), and choosing $\delta \leq \min\left\{1 - (1-\epsilon)^{\frac{1-p}{2}}, (1+\epsilon)^{\frac{1-p}{2}} - 1\right\}$ in (4.20), we conclude the proof. $\square$

Now we have done all the preparation work for proving Theorem 4.7.

Proof of Theorem 4.7. The proof consists of the following two steps, whose combination confirms the claim.

Step 1: Show that

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[\left|r^T-1\right| \wedge (N-1)\right] = 0, \tag{4.22}$$

where N is the constant appearing in (4.17). To this end, for a sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\left[\left|r^T-1\right| \wedge (N-1)\right] \\
&\quad = \mathbb{E}^{\mathbb{P}^T}\left[|r^T-1| \wedge (N-1) 1_{\{r^T\le N,\, |r^T-1|\le \epsilon\}}\right] + \mathbb{E}^{\mathbb{P}^T}\left[|r^T-1| \wedge (N-1) 1_{\{r^T\le N,\, |r^T-1|> \epsilon\}}\right] \\
&\qquad + \mathbb{E}^{\mathbb{P}^T}\left[|r^T-1| \wedge (N-1) 1_{\{r^T> N\}}\right] \\
&\quad \le \epsilon\, \mathbb{P}^T\left(r^T\le N, |r^T-1|\le\epsilon\right) + (N-1)\,\mathbb{P}^T\left(|r^T-1|>\epsilon\right) + (N-1)\,\mathbb{P}^T(r^T>N).
\end{aligned}$$

As $T\uparrow\infty$ in the previous inequalities, (4.22) follows from (4.17), (4.19), and the arbitrary choice of ϵ.

Step 2: Show that

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r^T-1|\, 1_{\{r^T>N\}}\right] = 0. \tag{4.23}$$

Note that $(r^T)^{1-p} > N^{1-p}$ on $\left\{r^T > N\right\}$. Recall that $\overline{\mathfrak{R}}N^{p-1} < 1$, thus

$$\left(1-\overline{\mathfrak{R}}N^{p-1}\right)\, \mathbb{E}^{\mathbb{P}^T}\left[|r^T-1|\, 1_{\{r^T>N\}}\right] \le \mathbb{E}^{\mathbb{P}^T}\left[\left|1-\frac{R\left(f^{1,T}\right)}{(r^T)^{1-p}}\right|\left|r^T-1\right|\right].$$

Now sending $T\uparrow\infty$, (4.23) follows from (4.16). □

4.4. *Convergence of Wealth Processes.* Now we switch from the convergence of optimal payoffs to the convergence of corresponding wealth processes. The following lemma bridges this transition.

Lemma 4.13. *Consider a sequence $\left\{Y^T\right\}_{T\in\mathbb{R}_+}$ of càdlàg processes and a sequence $\left\{\mathbb{P}^T\right\}_{T\in\mathbb{R}_+}$ of probability measures, such that:*

i) For each $T\in\mathbb{R}_+$, $Y^T_0 = 1$ and $Y^T_t > 0$ for all $t\in\mathbb{R}_+$, $\mathbb{P}^T$-a.s..
ii) Each Y^T is a $\mathbb{P}^T$-supermartingale.
iii) $\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}^T}\left[|Y^T_T-1|\right]=0$.

Then:

a) $\lim_{T\to\infty}\mathbb{P}^T\left(\sup_{u\in[0,T]}\left|Y^T_u-1\right|\ge\epsilon\right)=0$, for any $\epsilon>0$.
b) Define $L^T := \int_0^\cdot (1/Y^T_{t-})\, dY^T_t$, i.e., L^T is the stochastic logarithm of Y^T, for each $T\in\mathbb{R}_+$. Then $\lim_{T\to\infty}\mathbb{P}^T\left(\left[L^T,L^T\right]_T\ge\epsilon\right)=0$, for any $\epsilon>0$, where $[\cdot,\cdot]_T$ is the square bracket on $[0,T]$.

Proof. This result follows from Theorem 2.5 and Remark 2.6 in Kardaras (2010). (Note that Theorem 2.5 in Kardaras (2010) is stated under a fixed probability $\overline{\mathbb{P}}$ and on a fixed time interval $[0,T]$, but its proof remains valid for a sequence of probability measures $\left\{\mathbb{P}^T\right\}_{T\in\mathbb{R}_+}$ and a family of time intervals $\{[0,T]\}_{T\in\mathbb{R}_+}$.) □

The next Lemma shows that once wealth reaches zero, it must stay at zero.

LEMMA 4.14. *Let Assumptions 2.2 and 2.3 hold. Then* $\mathbb{P}(X_s = 0, X_t > 0) = 0$ *for all* $0 \le s < t \le T$ *and* $X \in \mathcal{X}^T$.

PROOF. By contradiction, suppose there exist $0 \le s < t \le T$ and $\xi \in \mathcal{X}^T$ such that the event $B := \{\xi_s = 0, \xi_t > 0\}$ is not $\mathbb{P}$-null. Let $\zeta \in \mathcal{X}^T$ be strictly positive and set $\xi^n = (1/(n+1))\zeta + (n/(n+1))\xi$ for all $n \in \mathbb{N}$; of course, ξ^n is strictly positive and belongs to $\mathcal{X}^T$ for all $n \in \mathbb{N}$. Now, let $A = \{\xi_s = 0\} \supseteq B$ and define $\tau = s1_A + T1_{\Omega \setminus A}$. Then, $\mathbb{P}(A) > 0$ and τ is a stopping time, so that $A = \{\tau < T\}$. Following ζ until τ, then switching to ξ^n at τ, and switching back to ζ at time t, construct $g^n \in \mathcal{X}^T$ such that

$$g^n_T = \zeta_T \left(1_{\Omega \setminus A} + \frac{\zeta_s}{\zeta_t} \frac{\xi^n_t}{\xi^n_s} 1_A \right) = \zeta_T \left(1_{\Omega \setminus A} + \frac{\zeta_s}{\zeta_t} \frac{\zeta_t + n\xi_t}{\zeta_s + n\xi_s} 1_A \right) \ge n\xi_t \frac{\zeta_T}{\zeta_t} 1_B,$$

where the last inequality follows from $B \subseteq A$ and $\xi_s 1_A = 0$. Now, suppose that $f \in \mathcal{X}^T$ maximizes expected power utility, and set $f^n = (1 - 1/\sqrt{n})f + (1/\sqrt{n})g^n$ for $n \in \mathbb{N}$. Since $\mathbb{E}^{\mathbb{P}}[U(\alpha f)] > -\infty$ holds for all $\alpha \in (0,1]$ (recall that U is a power utility and $\mathbb{E}^{\mathbb{P}}[U(f)] > -\infty$), and the inequality $f^n_T \ge (1 - 1/\sqrt{n})f + \sqrt{n}\xi_t(\zeta_T/\zeta_t)1_B$ is valid, where the right-hand side in a nondecreasing sequence in $n \in \mathbb{N}$, $\mathbb{E}^{\mathbb{P}}[U(f^n)] \le \mathbb{E}^{\mathbb{P}}[U(f)]$ for all $n \in \mathbb{N}$ holds only if $\mathbb{P}(B) = 0$. This results contradicts the existence of $\xi \in \mathcal{X}^T$ such that $\mathbb{P}(\xi_s = 0, \xi_t > 0) > 0$, completing the proof. □

Combining the previous lemmas with Theorems 4.5 and 4.7, Theorem 2.5 is proved as follows.

PROOF OF THEOREM 2.5. The statements follow from Lemma 4.13 directly, once we check that the assumptions of Lemma 4.13 are satisfied. First, $r^T_0 = 1$ since both investors have the same initial capital. Second, assuming $r^T_\cdot$ being a $\mathbb{P}^T$-supermartingale for a moment, then $r^T_t > 0$, $\mathbb{P}^T$-a.s., for any $t \le T$, because $r^T_T > 0$, $\mathbb{P}^T$-a.s. (see Corollary 4.2 and Remark 4.3). Third, $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r^T_T - 1|\right] = 0$ follows from Theorems 4.5 and 4.7. Hence it remains to show that $r^T_\cdot$ is a $\mathbb{P}^T$-supermartingale.

Combining Assumption 2.2 and the numéraire property, it follows that $\mathbb{E}^{\mathbb{P}^T}\left[X_T/X^{0,T}_T\right] \le 1$ for any wealth process X such that $X_T \in \mathcal{C}^T$. Then for any $s < t \le T$, it remains to show that:

$$\mathbb{E}^{\mathbb{P}^T}\left[X_t/X^{0,T}_t | \mathcal{F}_s\right] \le X_s/X^{0,T}_s. \tag{4.24}$$

Both denominators in above inequality are nonzero, since $X^{0,T}_T > 0$ and $\{X^{0,T}_s = 0\} \subseteq \{X^{0,T}_t = 0\}$ for any $s \le t$. To prove (4.24), fix any $A \in \mathcal{F}_s$, and construct the wealth process $\widetilde{X}$ via

$$\widetilde{X}_u := \begin{cases} X^{0,T}_u, & u \in [0,s) \\ X^{0,T}_s \frac{X_u}{X_s} 1_A + X^{0,T}_u 1_{\Omega\setminus A}, & u \in [s,t) \\ X^{0,T}_s \frac{X_t}{X_s} \frac{X^{0,T}_u}{X^{0,T}_t} 1_A + X^{0,T}_u 1_{\Omega\setminus A}, & u \in [t,T] \end{cases}.$$

The compounding property in Assumption 2.2 implies that $\widetilde{X}_T \in \mathcal{C}^T$. Noting that

$$\frac{\widetilde{X}_T}{X^{0,T}_T} = \frac{X^{0,T}_s}{X_s} \frac{X_t}{X^{0,T}_t} 1_A + 1_{\Omega\setminus A},$$

the claim follows from $\mathbb{E}^{\mathbb{P}^T}\left[\widetilde{X}_T/X^{0,T}_T\right] \le 1$ and the arbitrary choice of A. □

Proof of Corollary 2.7. First, note that $(\frac{d\mathbb{P}^T}{d\mathbb{P}}|_{\mathcal{F}_t})_{T\geq 0}$ is a constant sequence. Indeed, for any $t \leq T \leq S$,

$$\begin{aligned}
\left.\frac{d\mathbb{P}^S}{d\mathbb{P}}\right|_{\mathcal{F}_t} &= \frac{\mathbb{E}_t^{\mathbb{P}}\left[\left(X_S^{0,S}\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X_S^{0,S}\right)^p\right]} = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\left(X_S^{0,S}/X_T^{0,S}\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\left(X_S^{0,S}/X_T^{0,S}\right)^p\right]} = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\right]\mathbb{E}_t^{\mathbb{P}}\left[\left(X_S^{0,S}/X_T^{0,S}\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\right]\mathbb{E}^{\mathbb{P}}\left[\left(X_S^{0,S}/X_T^{0,S}\right)^p\right]}\\
&= \frac{\mathbb{E}_t^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X_T^{0,S}\right)^p\right]} = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\left(X_T^{0,T}\right)^p\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(X_T^{0,T}\right)^p\right]} = \left.\frac{d\mathbb{P}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t}.
\end{aligned}$$

Here, the third equality follows from the assumption that $X_T^{0,S}$ and $X_S^{0,S}/X_T^{0,S}$ are independent; the fourth equality holds since $X_S^{0,S}/X_T^{0,S}$ is independent of $\mathcal{F}_t$; and the fifth equality holds thanks to the myopic optimality $X_T^{0,T} = X_T^{0,S}$.

Second, note that $\frac{d\mathbb{P}^T}{d\mathbb{P}}$ is a strictly positive martingale; see the discussion after (2.2), it then induces a probability measure $\tilde{\mathbb{P}}$, which is equivalent to $\mathbb{P}$ on $\mathcal{F}_t$. As a result, the rest statements follows from Theorem 2.5 and Remark 2.6 part iii) directly. $\square$

5. Proof of the Turnpike for Diffusions. This section proves the turnpike theorem for diffusions and the explicit turnpike. First, v^T is constructed and its associated verification result is proved in Section 5.1. Next, for a fixed $t > 0$ and $y \in E$, the limiting behavior of $\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t}$ as $T \to \infty$ is studied in Section 5.2 . The proof of Proposition 2.15 shows that the family of densities $(\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t})_{T>0}$ converges to the long-run density $\frac{d\hat{\mathbb{P}}^y}{d\mathbb{P}^y}|_{\mathcal{F}_t}$ when the coordinate process of $(\hat{\mathbb{P}}^y)_{y\in E}$ is positive recurrent. Finally, Theorems 2.16 and 2.17 are proved in Section 5.3.

Remark 5.1. To ease notation, denote in the sequel:

$$\mathcal{E}\left(\int H\, dW\right)_{t,s} := \exp\left(\int_t^s H_u\, dW_u - \frac{1}{2}\int_t^s \|H_u\|^2\, du\right) \qquad \text{for any integrand } H \text{ and } t \leq s$$

and by $\mathcal{E}\left(\int H\, dW\right)_t = \mathcal{E}\left(\int H\, dW\right)_{0,t}$.

5.1. *Construction of v^T.* As a solution to (2.18), v^T is constructed via its long-run analogue $e^{\lambda_c(T-\cdot)}\hat{v}$. Let us first construct the long-run measure $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ introduced after Assumption 2.12.

Lemma 5.2. *Let Assumptions 2.8 and 2.12 hold*[6]*. Then:*

(i) There exists a unique solution $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ to the martingale problem for $\hat{\mathcal{L}}$ on $\mathbb{R}^d\times E$;
(ii) $\hat{\mathbb{P}}^\xi \sim \mathbb{P}^\xi$ for any $\xi \in \mathbb{R}^d\times E$.

Proof. Let Ω^k be the space of continuous maps from $[0,\infty)$ into $\mathbb{R}^k$, and $\mathcal{B}^k$ its Borel sigma algebra. Note that $(\Omega,\mathcal{B}) = (\Omega^{k+1},\mathcal{B}^{k+1})$. According to (Pinsky, 1995, Theorem 5.1.1), Assumption 2.12 part (2.11) ensures a solution $(\mathbb{P}_Y^y)_{y\in E} \in M_1(\Omega^1,\mathcal{B}^1)$ to the martingale problem for $\mathcal{L}^{\hat{v},0} = \mathcal{L} + A\frac{\hat{v}_y}{\hat{v}}$ on E, such that the coordinate process Y is recurrent in E. In particular $\mathbb{P}_Y^y(Y_t \in E, \forall t \geq 0) = 1$, for any $y \in E$. Therefore, with $(\mathbb{Q}^z)_{z\in\mathbb{R}^d}$ denoting Wiener measure on $(\Omega^d,\mathcal{B}^d)$, it follows that $\left(\mathbb{P}_{\hat{B},Y}^\xi := \mathbb{Q}^z\times\mathbb{P}_Y^y\right)_{\xi\in\mathbb{R}^d\times E}$ solves the martingale problem for $\mathcal{L}_{\hat{B},Y}$ on $\mathbb{R}^d\times E$, where:

$$\mathcal{L}_{\hat{B},Y} = \frac{1}{2}\sum_{i,j=1}^{d+1} A_{ij}^{\hat{B},Y}(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d+1} b_i^{\hat{B},Y}(\xi)\frac{\partial}{\partial\xi_i}, \quad A^{\hat{B},Y} = \begin{pmatrix} 1_d & 0\\ 0 & A\end{pmatrix}, \quad b^{\hat{B},Y} = \begin{pmatrix} 0\\ B + A\frac{\hat{v}_y}{\hat{v}}\end{pmatrix}.$$

[6]Condition (2.12) is not needed for this Lemma.

Set $\mathcal{F}_t = \mathcal{B}_{t+}$ for $t \geq 0$. Since $A^{\hat{B},Y}$ is non-degenerate and A satisfies Assumption 2.8, it follows that under $\mathbb{P}^{\xi}_{\hat{B},Y}$, there exist independent Wiener processes $\hat{W}$ and $\hat{B}$ such that $(\hat{B}, \hat{W})$ is a $d+1$ dimensional Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$. Defining $\hat{Z} = \rho\hat{W} + \bar{\rho}\hat{B}$ and

$$R_t := z + \int_0^t \frac{1}{1-p}(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}})(Y_s)ds + \int_0^t \sigma(Y_s)d\hat{Z}_s,$$

it follows that $\Big((R,Y),(\hat{Z},\hat{W}),(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}^{\xi}_{\hat{B},Y})\Big)$ solves (2.15). Moreover, it follows from Assumption 2.8 and $\mathbb{P}^y_Y(Y_t \in E, \forall t \geq 0) = 1$ that $\mathbb{P}^{\xi}_{\hat{B},Y}((R_t,Y_t) \in \mathbb{R}^d \times E, \forall t \geq 0) = 1$ for any $\xi \in \mathbb{R}^d \times E$.

Since weak solutions induce solutions to the martingale problem via Ito's formula, it follows that if $\hat{\mathbb{P}}^{\xi} \in M_1(\Omega,\mathcal{F})$ is defined by $\hat{\mathbb{P}}^{\xi}(A) = \mathbb{P}^{\xi}_{\hat{B},Y}((R,Y) \in A)$ with $A \in \mathcal{F}$, then $(\hat{\mathbb{P}}^{\xi})_{\xi\in\mathbb{R}^d\times E}$ solves the martingale problem for $\hat{\mathcal{L}}$. Hence, the SDE (2.15) holds for the coordinate process (R,Y) under $\hat{\mathbb{P}}^{\xi}$.

The statement in part (ii) follows from (Cheridito et al., 2005, Remark 2.6). Note that the assumption in Cheridito et al. (2005) is satisfied because of our Assumption 2.8, $\hat{v} > 0$ and $\hat{v} \in C^2(E)$ in Assumption 2.12, and the fact that $\mathbb{P}^{\xi}((R_t,Y_t) \in \mathbb{R}^d \times E, \forall t \geq 0) = \hat{\mathbb{P}}^{\xi}((R_t,Y_t) \in \mathbb{R}^d \times E, \forall t \geq 0) = 1$ for any $\xi \in \mathbb{R}^d \times E$. □

This is the proof for a sufficient condition for Assumption 2.12.

Proof of Proposition 2.13. By Theorem 18 in Guasoni and Robertson (2009), under Assumptions 2.8, 2.10, and 2.11, (2.17) yields the existence of a function $\hat{v}$ which satisfies (2.10), (2.11), along with the first inequality in (2.12). By Holder's inequality, (2.16) ensures that the second inequality in (2.12) holds as well, proving the assertion. □

Recall that $\hat{\mathbb{P}}^{\xi}$ is denoted by $\hat{\mathbb{P}}^y$ for $\xi = (0,y)$. Now, to construct the solution v^T to (2.18), introduce the auxiliary function h^T, defined as:

$$h^T(t,y) := \mathbb{E}^{\hat{\mathbb{P}}^y}\left[(\hat{v}(Y_{T-t}))^{-1}\right], \quad \text{for } (t,y) \in [0,T] \times E. \tag{5.1}$$

Then, the candidate reduced value function v^T is:

$$v^T(t,y) := e^{\lambda_c(T-t)}\hat{v}(y)h^T(t,y). \tag{5.2}$$

Thus, h^T is the ratio between v^T and its long-run analogue $e^{\lambda_c(T-\cdot)}\hat{v}$. The verification result Proposition 5.5 below confirms that v^T is a strictly positive classical solution to (2.18) and the relation $u^T(t,x,y) = \frac{x^p}{p}(v^T(t,y))^{\delta}$ holds for $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times E$.

The next Proposition characterizes the function h^T. Clearly, $h^T(t,y) > 0$ for $(t,y) \in [0,T] \times E$.

Proposition 5.3. *Let Assumptions 2.8, 2.10, 2.11 and 2.12 hold*[7]*. Then $h^T(t,y) < \infty$ for all $(t,y) \in [0,T] \times E$, $h^T(t,y) \in C^{1,2}((0,T) \times E)$, and h^T satisfies*

$$\begin{aligned} &\partial_t h^T + \mathcal{L}^{\hat{v},0}h^T = 0, \quad (t,y) \in (0,T) \times E, \\ &h^T(T,y) = \frac{1}{\hat{v}(y)}, \quad y \in E, \end{aligned} \tag{5.3}$$

where $\mathcal{L}^{\hat{v},0} := \mathcal{L} + A\frac{\hat{v}_y}{\hat{v}}\partial_y$.

[7]Condition (2.12) is not needed for this Proposition.

Proof. To see that $h^T(t,y) < \infty$ for all $(t,y) \in [0,T] \times E$, note that by (2.14) the generator for Y under $\hat{\mathbb{P}}^y$ is $\mathcal{L}^{\hat{v},0} = \mathcal{L} + A\frac{\hat{v}_y}{\hat{v}}\partial_y$. It follows from (2.10) that

$$\mathcal{L}^{\hat{v},0}\left((\hat{v})^{-1}\right) = (c - \lambda_c)(\hat{v})^{-1} \quad \text{on } E. \tag{5.4}$$

Combining the previous equation with $\sup_{y\in E} c(y) < \infty$ in Assumption 2.11, we can find $K > 0$ such that

$$\left(\partial_t + \mathcal{L}^{\hat{v},0}\right)\left(\frac{e^{-Kt}}{\hat{v}(y)}\right) \le 0 \quad \text{on } [0,T] \times E.$$

Thus, using the strict positivity of $\hat{v}$, Assumption 2.8 and Fatou's lemma, it follows that:

$$h^T(t,y) = \mathbb{E}^{\hat{\mathbb{P}}}\left[(\hat{v}(Y_T))^{-1}\,\middle|Y_t = y\right] \le \frac{e^{K(T-t)}}{\hat{v}(y)} < \infty. \tag{5.5}$$

It is next shown that $h^T \in C^{1,2}((0,T)\times E)$ satisfies (5.3). To this end, the classical version of the Feynman-Kac formula (see e.g. Theorem 5.3 in Friedman (1975) pp. 148) does not apply directly because a) the operator $\mathcal{L}^{\hat{v},0}$ is not assumed to be uniformly elliptic on E, and b) $(\hat{v})^{-1}$ may grow faster than polynomial near the boundary of E. Rather, the statement is proved using Theorem 1 in Heath and Schweizer (2000), which yields that h^T is a classical solution of (5.3).

To check that the assumptions of Theorem 1 in Heath and Schweizer (2000) are satisfied, note that, since A is locally Lipschitz on E due to Assumption 2.8, Lemma 1.1 in Friedman (1975) pp. 128 implies that a is also locally Lipschitz on E. On the other hand, the local Lipschitz continuity of $B + A\frac{\hat{v}_y}{\hat{v}}$ is ensured by Assumption 2.8 and $\hat{v} \in C^2(E)$. Hence (A1) in Heath and Schweizer (2000) is satisfied. Second, (A2) in Heath and Schweizer (2000) holds thanks to the well-posedness of the martingale problem $(\hat{\mathbb{P}})$, in particular the coordinate process Y does not hit the boundary of E under $\hat{\mathbb{P}}$. Third, (A3') in Heath and Schweizer (2000), (A3a')-(A3d') are clearly satisfied under our assumptions.

In order to check (A3e'), it suffices to show that h^T is continuous in any compact sub-domain of $(0,T)\times E$. To this end, recall that the domain is $E = (\alpha,\beta)$ for $-\infty \le \alpha < \beta \le \infty$. Let $\{\alpha_m\}$ and $\{\beta_m\}$ two sequences such that $\alpha_m < \beta_m$ for all m, α_m strictly decreases to α, and β_m strictly increases to β. Set $E_m = (\alpha_m, \beta_m)$. For each m there exists a function $\psi_m(y) \in C^\infty(E)$ such that a) $\psi_m(y) \le 1$, b) $\psi_m(y) = 1$ on E_m, and c) $\psi_m(y) = 0$ on $E \cap E^c_{m+1}$. To construct such ψ_m let $\varepsilon_m = \frac{1}{3}\min\{\beta_{m+1} - \beta_m, \alpha_m - \alpha_{m+1}\}$ and then take

$$\psi_m(y) = \eta_{\varepsilon_m} * 1_{\{\alpha_m - \varepsilon_m, \beta_m + \varepsilon_m\}}(y),$$

where η_{ε_m} is the standard mollifier and $*$ is the convolution operator. Define the functions f_m and $h^{T,m}$ by

$$f_m(y) = \frac{\psi_m(y)}{\hat{v}(y)} \quad \text{and} \quad h^{T,m}(t,y) = \mathbb{E}^{\hat{P}^y}\left[f_m(Y_{T-t})\right].$$

By construction, for all $y \in E$, $\uparrow \lim_{m\uparrow\infty} f_m(y) = (\hat{v}(y))^{-1}$. It then follows from the monotone convergence theorem and (5.5) that $\lim_{m\uparrow\infty} h^{T,m}(t,y) = h^T(t,y)$.

Since $\hat{v} \in C^2(E)$ and $\hat{v} > 0$, each $f_m(y) \in C^2(E)$ is bounded. It then follows from the Feller property for $\hat{\mathbb{P}}^y$ (see Theorem 1.13.1 in Pinsky (1995)) that $h^{T,m}$ is continuous in y. On the other hand, by construction of f_m and (5.4), there exists a constant $K_m > 0$ such that

$$a|\dot{f}_m| \le K_m, \qquad \left|\mathcal{L}^{\hat{v},0} f_m\right| \le K_m, \quad \text{on } E. \tag{5.6}$$

Moreover, Ito's formula gives that, for any $0 \leq s \leq t \leq T$,

$$f_m(Y_t) = f_m(Y_s) + \int_s^t \mathcal{L}^{\hat{v},0} f_m(Y_u)\,du + \int_s^t a\dot{f}_n(Y_u)\,d\hat{W}_u.$$

Combining the previous equation with estimates in (5.6), it follows that:

$$\sup_{y\in E} \left| \mathbb{E}^{\hat{\mathbb{P}}^y}\left[f_m(Y_t) - f_m(Y_s)\right]\right| \leq K_m(t-s).$$

Therefore, $h^{T,m}$ is uniformly continuous in t. Combining with the continuity of $h^{T,m}$ in y, we conclude that $h^{T,m}$ is jointly continuous in (t,y) on $[0,T]\times E$.

Note that the operator $\mathcal{L}^{\hat{v},0}$ is uniformly elliptic in the parabolic domain $(0,T)\times E_m$. It then follows from a straightforward calculation that $h^{T,m}$ satisfies the differential equation:

$$\partial_t h^{T,m} + \mathcal{L}^{\hat{v},0} h^{T,m} = 0 \quad (t,y)\in(0,T)\times E_m.$$

Note that $(h^{T,m})_{m\geq 0}$ is uniformly bounded from above by h^T, which is finite on $[0,T]\times E_m$. Appealing to the *interior Schauder estimate* (see e.g. Theorem 15 in Friedman (1964) pp. 80), there exists a subsequence $(h^{T,m'})_{m'}$ which converges to h^T uniformly in $(0,T)\times D$ for any compact sub-domain D of E_m. Since each $h^{T,m'}$ is continuous and the convergence is uniform, we confirm that h^T is continuous in $(0,T)\times D$. Since the choice of D is arbitrary in E_m, (A3e') in Heath and Schweizer (2000) is satisfied. □

We prepare for the verification result in Proposition 5.5 by introducing some notation. Let Assumption 2.8 and 2.10 hold. For any $w(t,y)\,:\,[0,T]\times E$ which is strictly positive and in $C^{1,2}((0,T)\times E)$, define the process

$$D^w_t := \mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{w_y}{w}\right)'\frac{1}{a}dW - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{w_y}{w}\right)'\sigma\bar{\rho}dB\right)_t, \tag{5.7}$$

where (B,W) is a $d+1$ dimensional $(\mathbb{P}^y,(\mathcal{F}_t)_{t\geq 0})$ Wiener process. For $t\leq s\leq T$ define $D^w_{t,s}$ as in Remark 5.1. Set the portfolio

$$\pi^w = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{w_y}{w}\right) \tag{5.8}$$

evaluated at (t,Y_t) and let X^{π^w} denote the wealth process. Set $\eta^w = \delta\frac{w_y}{w}$ and define the process M^{η^w} via

(5.9)

$$M^{\eta^w}_t = e^{-\int_0^t r dt}\mathcal{E}\left(\int\left(-\Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)\eta^w\right)'\frac{1}{a}dW - \int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\eta^w\right)'\sigma\bar{\rho}dB\right)_t.$$

The following Proposition plays a central role to verify the optimality of both finite horizon and long-run optimal portfolios as well as comparing their terminal wealths. A similar statement for the long-run limit is proved in (Guasoni and Robertson, 2009, Theorem 7).

Proposition 5.4. *Let Assumptions 2.8 and 2.10 hold. Assume there exists a function w : $[0,T]\times E\to\mathbb{R}$ and a constant $\lambda\in\mathbb{R}$, such that $w\in C^{1,2}((0,T)\times E,\mathbb{R})$ is strictly positive and satisfies the differential expression*

$$\partial_t w + \mathcal{L}w + (c-\lambda)w = 0, \quad (t,y)\in(0,T)\times E.$$

Then the following conclusions hold:

(i) For all admissible portfolios π and all $y \in E$, the process $X^\pi M^{\eta^w}$ is a non-negative supermartingale under $\mathbb{P}^y$ where M^{η^w} is given in (5.9).

(ii) For all $t \leq s \leq T$ and $y \in E$ the processes X^{π^w} and M^{η^w} satisfy the $\mathbb{P}^y$ almost sure identities

$$(5.10)\qquad \begin{aligned} \left(X_s^{\pi^w}\right)^p &= \left(X_t^{\pi^w}\right)^p \left(w(t,Y_t)e^{\lambda(s-t)}\right)^\delta D^w_{t,s} w(s,Y_s)^{-\delta}, \\ \left(M_s^{\eta^w}\right)^q &= \left(M_t^{\eta^w}\right)^q \left(w(t,Y_t)e^{\lambda(s-t)}\right)^{\frac{\delta}{1-p}} D^w_{t,s} w(s,Y_s)^{-\frac{\delta}{1-p}}, \end{aligned}$$

where π^w, M^{η^w} and D^w are as in (5.8), (5.9) *and* (5.7) *respectively.*

Proof. Given upon w, it is clear, using stochastic integration by parts, that for $i = 1, \dots, d$ the process $M^{\eta^w} S^i$ is a non-negative supermartingale under $\mathbb{P}^y$ for any $y \in E$. Thus, part (i) follows.

It remains the show the almost-sure identities. To this end fix $t \leq s \leq T$. By (2.8) it follows that

$$(5.11)\qquad \begin{aligned} \frac{\left(X_s^{\pi^w}\right)^p}{\left(X_t^{\pi^w}\right)^p} &= \exp\left(\int_t^s \left(p\mu'\pi^w + pr - \frac{p}{2}(\pi^w)'\Sigma\pi^w\right) d\tau + p\int_t^s (\pi^w)'\sigma dZ_\tau\right), \\ \frac{\left(M_s^{\eta^w}\right)^q}{\left(M_t^{\eta^w}\right)^q} &= e^{-q\int_t^s r d\tau} \mathcal{E}\left(\int \left(-\Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)\eta^w\right)' \frac{1}{a} dW - \int \left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\eta^w\right)' \sigma\bar{\rho} dB\right)^q_{t,s}. \end{aligned}$$

To prove the first equality, it suffices to show, by taking logarithms and expanding $Z = \rho W + \bar{\rho} B$ that

$$(5.12)\qquad \begin{aligned} &\int_t^s \left(p\mu'\pi^w + pr - \frac{p}{2}(\pi^w)'\Sigma\pi^w\right) d\tau + p\int_t^s (\pi^w)'\sigma\rho dW_\tau + p\int_t^s (\pi^w)'\sigma\bar{\rho} dB_\tau \\ &\qquad = \delta \log w(t,Y_t) + \delta\lambda(s-t) + \log D^w_{t,s} - \delta \log w(s,Y_s). \end{aligned}$$

Multidimensional notation makes calculations more transparent. Set $\omega := \delta \log w$. Then ω solves the semi-linear differential expression

$$\partial_t \omega + \mathcal{L}\omega + \frac{1}{2}\nabla\omega' \left(A - q\Upsilon'\Sigma^{-1}\Upsilon\right) \nabla\omega + \delta(c - \lambda) = 0, \quad (t,y) \in (0,T) \times E,$$

because $\delta^{-1}A = A - q\Upsilon'\Sigma^{-1}\Upsilon$. Since $\mathcal{L} = \frac{1}{2}A\partial_{yy} + (b - q\Upsilon'\Sigma^{-1}\mu)\partial_y$, Ito's formula implies that *under* $\mathbb{P}^y$ (see (2.4)), and expanding $c = \frac{1}{\delta}\left(pr - \frac{1}{2}q\mu'\Sigma^{-1}\mu\right)$:

$$(5.13)\qquad \begin{aligned} &\delta \log w(s,Y_s) - \delta \log w(t,Y_t) = \\ &\int_t^s \left(q\mu'\Sigma^{-1}\Upsilon\nabla\omega - \frac{1}{2}\nabla\omega'(A - q\Upsilon'\Sigma^{-1}\Upsilon)\nabla\omega + \delta\lambda - \left(pr - \frac{1}{2}q\mu'\Sigma^{-1}\mu\right)\right) d\tau + \int_t^s \nabla\omega' a dW_\tau. \end{aligned}$$

Again, using $\delta^{-1}A = A - q\Upsilon'\Sigma^{-1}\Upsilon$ and noting that $\delta\frac{w_y}{w} = \omega_y$, after some simplifications it follows that:

$$(5.14)\qquad \begin{aligned} \log D^w_{t,s} &= \int_t^s \left(-q\mu'\Sigma^{-1}\Upsilon + \nabla\omega'(A - q\Upsilon'\Sigma^{-1}\Upsilon)\right) \frac{1}{a} dW \\ &\quad - q\int_t^s \left(\mu'\Sigma^{-1} + \nabla\omega\Upsilon'\Sigma^{-1}\right) \sigma\bar{\rho} dB \\ &\quad + \int_t^s \left(-\frac{1}{2}q^2\mu'\Sigma^{-1}\mu + q(1-q)\mu'\Sigma^{-1}\Upsilon\nabla\omega - \frac{1}{2}\nabla\omega'\left(A - (2q - q^2)\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla\omega\right) d\tau. \end{aligned}$$

Lastly, plugging in for π^w yields

$$
\begin{aligned}
&\int_t^s \left(p\mu'\pi^w + pr - \frac{p}{2}(\pi^w)'\Sigma\pi^w\right) d\tau + p\int_t^s (\pi^w)'\sigma\rho dW_\tau + p\int_t^s (\pi^w)'\sigma\bar{\rho} dB_\tau \\
&= \int_t^s \left(pr - \frac{1}{2}q(1+q)\mu'\Sigma^{-1}\mu - q^2\mu'\Sigma^{-1}\Upsilon\nabla\omega - \frac{1}{2}q(q-1)\nabla\omega'\Upsilon'\Sigma^{-1}\Upsilon\nabla\omega\right) d\tau \\
&\quad - q\int_t^s \left(\mu' + \nabla\omega'\Upsilon'\right)\Sigma^{-1}\sigma\rho dW_\tau - q\int_t^s \left(\mu' + \nabla\omega'\Upsilon'\right)\Sigma^{-1}\sigma\bar{\rho} dB_\tau.
\end{aligned} \tag{5.15}
$$

Now, using (5.13), (5.14) and (5.15), the equality in (5.12) follows by matching the respective $d\tau$, dW and dB terms.

The proof for the second identity in (5.10) is similar. Given (5.11), it suffices to show that, by taking logarithms

$$
\begin{aligned}
&\int_t^s r d\tau + \log \mathcal{E}\left(\int \left(-\Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)\eta^w\right)\frac{1}{a} dW - \int \left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\eta^w\right)\sigma\bar{\rho} dB\right)_{t,s} \\
&\qquad = \frac{\delta}{1-p}\log w(t, Y_t) + \frac{\delta}{1-p}\lambda(s-t) + \log D^w_{t,s} - \frac{\delta}{1-p}\log w(s, Y_s).
\end{aligned} \tag{5.16}
$$

The equality in (5.13) (multiplied by $\frac{1}{1-p}$), combined with that in (5.14) yield an expression for the right hand side of the above equation in terms of integrals from s to t of $d\tau, dW$ and dB. As for the left hand side, a lengthy calculation shows that

$$
\begin{aligned}
&\int_t^s qr d\tau + q\log \mathcal{E}\left(\int \left(-\Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)\eta^w\right)'\frac{1}{a} dW - \int \left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\eta^w\right)'\sigma\bar{\rho} dB\right)_{t,s} \\
&= \int_t^s \left(-qr - \frac{1}{2}q\mu'\Sigma^{-1}\mu - \frac{1}{2}q\nabla\omega'(A - \Upsilon'\Sigma^{-1}\Upsilon)\nabla\omega\right) d\tau \\
&\quad + q\int_t^s \left(-\mu'\Sigma^{-1}\Upsilon + \nabla\omega'(A - \Upsilon'\Sigma^{-1}\Upsilon)\right)\frac{1}{a} dW_\tau \\
&\quad - q\int_t^2 \left(\mu'\Sigma^{-1} + \nabla\omega'\Upsilon'\Sigma^{-1}\right)\sigma\bar{\rho} dB_\tau.
\end{aligned} \tag{5.17}
$$

Thus, using (5.13), (5.14) and (5.17), the equality in (5.16) follows by matching $d\tau, dW$ and dB terms. □

Now we are ready to state the verification result for the finite horizon problem.

Proposition 5.5. *Let Assumptions 2.8, 2.10, 2.11 and Assumption 2.12 hold*[8]*. Then:*

(i) v^T *has the stochastic representation:*

$$
v^T(t, y) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^T c(Y_s) ds\right) \big| Y_t = y\right], \qquad \text{for } (t, y) \in [0, T] \times E. \tag{5.18}
$$

(ii) $v^T > 0$, $v^T \in C^{1,2}((0, T) \times E)$, *and it solves* (2.18).

[8]Condition (2.12) is not needed for this Proposition.

(iii) $u^T(t,x,y) = \frac{x^p}{p}\left(v^T(t,y)\right)^\delta$ *on* $[0,T]\times\mathbb{R}_+\times E$ *and* π^T *in* (2.19) *is the optimal portfolio.*

REMARK 5.6. The representation (5.18) has been shown in Remark 3.4 of Zariphopoulou (2001). We provide a novel proof which ties to the h-transform (Pinsky, 1995, Chapter 4) of $\mathcal{L}+c$.

PROOF. Items (i) and (ii) are proved first. Given that h^T solves (5.3), long but straightforward calculations using (2.10) show that v^T solves (2.18). Moreover, $v^T \in C^{1,2}((0,T)\times E)$ because $\hat{v}\in C^2(E)$ and $h^T \in C^{1,2}((0,T)\times E)$.

To prove the stochastic representation in (5.18), define the operator:

$$\mathcal{L}^{\hat{v}} := \mathcal{L}^{\hat{v},0} + \lambda_c.$$

where $\mathcal{L}^{\hat{v},0}$ is the same as in Proposition 5.3. It is the h-transform of $\mathcal{L}+c$ with $h=\hat{v}$ (Pinsky, 1995, Chapter 4.2). Following the notation in (Pinsky, 1995, Chapter 4.0), define the transition measure $\hat{p}(t,y,dz)$ of $\mathcal{L}^{\hat{v}}$ on E by

$$\hat{p}(s,y,B) := \mathbb{E}^{\hat{\mathbb{P}}^y}\left[e^{\lambda_c s}\, 1_B\right], \qquad \text{for any measurable } B\subseteq E; \tag{5.19}$$

and the transition measure $p(t,y,dz)$ for the operator $\mathcal{L}+c$ on E by

$$p(s,y,B) := \mathbb{E}^{\mathbb{P}^y}\left[\exp\left(\int_0^s c(Y_s)ds\right) 1_B\right], \qquad \text{for any measurable } B\subseteq E, \tag{5.20}$$

where $c(y)$ is from Assumption 2.11. These two transition measures are related by the following identity (Pinsky, 1995, Theorem 4.1.1):

$$\hat{v}(y)\hat{p}(s,y,dz)\frac{1}{\hat{v}(z)} = p(s,y,dz), \qquad \text{for any } s\geq 0, y,z\in E. \tag{5.21}$$

Thus, choosing $s=T-t$ in (5.21) and integrating both sides on E, the Markov property implies that:

$$v^T(y,t) = e^{\lambda_c(T-t)}\hat{v}(y)\,\mathbb{E}^{\hat{\mathbb{P}}^y}\left[(\hat{v}(Y_T))^{-1}\,|Y_t=y\right] = \mathbb{E}^{\mathbb{P}^y}\left[\exp\left(\int_t^T c(Y_s)ds\right)|Y_t=y\right],$$

where the first identity follows from (5.1) and the definition of v^T in (5.2). This proves (5.18). Given (5.18) the strict positivity of v^T follows immediately.

It remains to show the verification result in item *(iii)*. By applying Proposition 5.4 to $w = v^T, \lambda=0$ it follows by evaluating (5.10) at $t=t, s=T$ that for the portfolio in (2.19) and the process $M^{\eta^{v^T}}$ from (5.9)

$$\mathbb{E}^{\mathbb{P}^y}\left[\frac{1}{p}\left(X_T^{\pi^T}\right)^p|X_t=x,Y_t=y\right] = \frac{x^p}{p}\left(v^T(t,y)\right)^\delta\mathbb{E}^{\mathbb{P}^y}\left[D_{t,T}^{v^T}|X_t=x,Y_t=y\right], \tag{5.22}$$

since $v^T(T,y)=1$. In a similar manner

$$\frac{x^p}{p}\mathbb{E}^{\mathbb{P}^y}\left[\left(M_T^{\eta^{v^T}}\right)^q|X_t=x,Y_t=y\right]^{1-p} = \frac{x^p}{p}\left(v^T(t,y)\right)^\delta\mathbb{E}^{\mathbb{P}^y}\left[D_{t,T}^{v^T}|X_t=x,Y_t=y\right]^{1-p}.$$

Therefore, thanks to duality results for power utility between payoffs and stochastic discount factors (Guasoni and Robertson, 2009, Lemma 5), the claims in part *(iii)* of Proposition 5.5 will follow if it can be shown that D^{v^T} is a $\mathbb{P}^y$ martingale for all $y\in E$.

To this end, note that for $w = \hat{v}$ the process of (5.7) is precisely the stochastic exponential which changes the dynamics from $\mathbb{P}^y$ to those for $\hat{\mathbb{P}}^y$. It follows from part (ii) of Lemma 5.2 and the backward martingale theorem (Cheridito et al., 2005, Remark 2.3.2) that $D^{\hat{v}}$ is a $(\hat{\mathbb{P}}^y, (\mathcal{F}_t)_{t\geq 0})$ martingale. Therefore, it follows that

$$\left.\frac{d\hat{\mathbb{P}}^y}{d\mathbb{P}^y}\right|_{\mathcal{F}_t} = D_t^{\hat{v}}. \tag{5.23}$$

Furthermore, the Brownian motions $\hat{B}, \hat{W}$ from (2.15) are related to B, W by

$$\begin{aligned}\hat{B} &= B + q\bar{\rho}\sigma'\Sigma^{-1}\mu + q\Delta' a\frac{\hat{v}_y}{\hat{v}},\\ \hat{W} &= W + q\rho\sigma'\Sigma^{-1}\mu - a\frac{\hat{v}_y}{\hat{v}},\end{aligned}$$

where $\Delta = q\delta\rho'\bar{\rho}$. The dynamics of Y under $\hat{\mathbb{P}}^y$ are given in (2.15). Using the differential expression for h in (5.3), Ito's formula implies for all t that

$$\frac{h^T(t, Y_t)}{h^T(0, y)} = \mathcal{E}\left(\int a\frac{h_y^T}{h^T}d\hat{W}_s\right)_t. \tag{5.24}$$

It follows from (5.2) that

$$\frac{v_y^T}{v^T} = \frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}. \tag{5.25}$$

Now, using (5.25), D^{v^T} specifies to

(5.26)
$$\begin{aligned}D_t^{v^T} &= \mathcal{E}\left(\int \left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{v_y^T}{v^T}\right)'\frac{1}{a}dW - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{v_y^T}{v^T}\right)'\sigma\bar{\rho}dB\right)_t\\
&=\mathcal{E}\left(\int \left(-q\Upsilon'\Sigma^{-1}\mu + A\left(\frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}\right)\right)'\frac{1}{a}dW - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\left(\frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}\right)\right)'\sigma\bar{\rho}dB\right)_t\\
&=\mathcal{E}\left(\int \left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{\hat{v}_y}{\hat{v}}\right)'\frac{1}{a}dW - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{\hat{v}_y}{\hat{v}}'\sigma\bar{\rho}\right)dB\right)_t\\
&\quad\mathcal{E}\left(\int a\frac{h_y^T}{h^T}d\hat{W}_t - \int a\frac{h_y^T}{h^T}\Delta' d\hat{B}\right)_t\\
&=D_t^{\hat{v}}\mathcal{E}\left(\int a\frac{h_y^T}{h^T}d\hat{W}_t - \int a\frac{h_y^T}{h^T}\Delta' d\hat{B}\right)_t.\end{aligned}$$

The second to last equality follows from the identity for any processes a, b and Wiener process W that

$$\mathcal{E}\left(\int (a_s + b_s)dW_s\right) = \mathcal{E}\left(\int a_s dW_s\right)\mathcal{E}\left(\int b_s dW_s - \int b_s a_s ds\right).$$

It follows from Proposition 5.3 that $0 < \mathbb{E}^{\hat{\mathbb{P}}^y}\left[h^T(T, Y_T)\right] = h^T(0, y) < \infty$. Then the right hand side of (5.24) is a strictly positive $\hat{\mathbb{P}}^y$ martingale on $[0, T]$. The continuity assumptions upon A, h^T

imply $\int_0^t A(h_y^T/h^T)^2(t,Y_t)\,dt<\infty, \hat{\mathbb{P}}^y-a.s.$ Thus, since $Y,\hat{B}$ are independent we can apply Lemma 4.8 in Karatzas and Kardaras (2007) to conclude that

$$\mathcal{E}\left(\int a\frac{h_y^T}{h^T}d\hat{W}_t-\int a\Delta\frac{h_y^T}{h^T}d\hat{B}\right).$$

is also a strictly positive $\hat{P}^y$ martingale on $[0,T]$. Therefore, using (5.26) it follows that $\mathbb{E}^{\mathbb{P}^y}[D_t^{v^T}]=1$ and hence D^{v^T} is a strictly positive $\mathbb{P}^y$ martingale. This finishes the proof. □

5.2. *Convergence of conditional densities and wealth processes.* Before proving Proposition 2.15, it is necessary to relate the terminal wealths resulting from using the finite horizon optimal strategies π^T of (2.19) and the long-run optimal strategy $\hat{\pi}$ of (2.20).

Set $D^T:=D^{v^T}$. Proposition 5.5 implies that, for $w=v^T$ and $\lambda=0$ the hypotheses of Proposition 5.4 hold. Therefore, (5.10) implies that for π^T as in (2.19) and any $0\le t\le T$, under $\hat{\mathbb{P}}^y$ the corresponding wealth $X_t^{\pi^T}$ satisfies (recall that $\mathbb{P}^y$ and $\hat{\mathbb{P}}^y$ are equivalent on $\mathcal{F}_t$)

$$\left(X_t^{\pi^T}\right)^p=x^pD_t^T\left(\frac{v^T(0,y)}{v^T(t,Y_t)}\right)^\delta=x^pD_t^Te^{\delta\lambda_ct}\left(\frac{\hat{v}(y)h^T(0,y)}{\hat{v}(Y_t)h^T(t,Y_t)}\right)^\delta, \tag{5.27}$$

where we have used (5.2). In a similar manner, Assumption 2.12 implies that, for $w=\hat{v}$ and $\lambda=\lambda_c$, the hypothesis of Proposition 5.4 hold. Therefore, with $\hat{D}:=D^{\hat{v}}$ it follows by (5.10) that under $\hat{\mathbb{P}}^y$, for the long-run optimal strategy $\hat{\pi}$ defined in (2.20), the corresponding wealth process satisfies for any $0\le t\le T$

$$\left(X_t^{\hat{\pi}}\right)^p=x^p\hat{D}_te^{\delta\lambda_ct}\left(\frac{\hat{v}(y)}{\hat{v}(Y_t)}\right)^\delta. \tag{5.28}$$

By (5.26) and (5.24) it follows that

$$\frac{D_t^T}{\hat{D}_t}=\frac{h^T(t,Y_t)}{h^T(0,y)}\,\mathcal{E}\left(-\int a\frac{h_y^T}{h^T}\Delta' d\hat{B}\right)_t. \tag{5.29}$$

Therefore, (5.27), (5.28) and (5.29) imply

$$\frac{X_t^{\pi^T}}{X_t^{\hat{\pi}}}=\left(\frac{D_t^T}{\hat{D}_t}\right)^{1/p}\left(\frac{h^T(t,Y_t)}{h^T(0,y)}\right)^{-\delta/p}=\left(\frac{h^T(t,Y_t)}{h^T(0,y)}\right)^{\frac{1-\delta}{p}}\mathcal{E}\left(-\int a\Delta\frac{h_y^T}{h^T}\,d\hat{B}\right)_t^{\frac{1}{p}}. \tag{5.30}$$

where the last equality uses (5.29).

The proof of Proposition 5.5 showed D^T is a

$\mathbb{P}^y$ martingale for each $y\in E$. Thus, (5.22) and (5.27) in conjunction with (2.2) implies that $D_t^T=\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t}$. Similarly, as shown in (5.23), $\hat{D}_t=\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t}$. Therefore, equations (5.29) and (5.30) will be used to study the long horizon ($T\to\infty$) behavior of the ratios between wealth processes $X_t^{\pi^T}$ and $X_t^{\hat{\pi}}$ of the finite horizon and long run optimal portfolios, and their corresponding density processes $d\mathbb{P}^T/d\mathbb{P}|_{\mathcal{F}_t}$ and $d\hat{\mathbb{P}}^y/d\mathbb{P}|_{\mathcal{F}_t}$.

In order to prove Theorem 2.15, in light of (5.29) it is important to understand limit of h^T as $T\to\infty$. Under (2.12), an ergodic argument yields the existence of a strictly positive constant K such that:

$$\lim_{T\to\infty}h^T(t,y)=K,\qquad \text{for all } (t,y)\in\mathbb{R}_+\times E.$$

Developing this argument requires some terminology from ergodic theory. Consider a second order elliptic operator $L = \frac{1}{2}a_{ij}\partial^2_{ij} + b_i\,\partial_i$ on $\mathcal{D} \subseteq \mathbb{R}^n$. Let L^* denote its adjoint operator (if it exists). Suppose the martingale problem for the operator L is well posed on $\mathcal{D}$, and denote its solution by $(\mathbb{P}^x)_{x\in\mathcal{D}}$, with coordinate process Ξ. In the language of ergodic theory, Ξ is *recurrent*, if $\mathbb{P}^x(\tau(\epsilon, y) < \infty) = 1$ for any $(x, y) \in \mathcal{D}^2$ and $\epsilon > 0$, where $\tau(\epsilon, y) = \inf\{t \geq 0 \,|\, |\Xi_t - y| \leq \epsilon\}$. If Ξ is recurrent, Theorem 4.3.3 (i) in Pinsky (1995) implies that L is *critical*; see the Definition in pp. 145 of Pinsky (1995). Then there exists a unique solution φ (up to a constant multiple) of $L\varphi = 0$. This solution is called the *ground state* of L; see the Definition in pp. 149 of Pinsky (1995). For the L given above, its ground state is $\varphi = 1$. On the other hand, Theorem 4.3.3 (v) in Pinsky (1995) shows that L is critical if and only if L^* is critical. Then, denote the ground state of L^* by φ^*. Now if Ξ is recurrent, it is *positive recurrent*, or *ergodic*, if $\int_{\mathcal{D}} \varphi^*(y)\,dy < \infty$, and *null recurrent* otherwise.

Now we are ready the show the limiting behavior of h^T as $T \uparrow \infty$.

Proposition 5.7. *Let Assumptions 2.8 and 2.12 hold. Then:*

$$\lim_{T\to\infty} h^T(t,y) = \frac{\int_\alpha^\beta \hat{v}\, m(y)\,dy}{\int_\alpha^\beta \hat{v}^2\, m(y)\,dy} =: K, \qquad \textit{for any } (t,y) \in \mathbb{R}_+ \times E.$$

Proof. Consider the second order operator $\mathcal{L}^{\hat{v},0}$ from Proposition 5.3. As shown in Lemma 5.2, (2.11) in Assumption 2.12 ensures that the martingale problem for $\mathcal{L}^{\hat{v},0}$ on E is well-posed, and Y denotes its coordinate process. It then follows from Corollary 5.1.11 in Pinsky (1995) that Y is positive recurrent if and only if the first integral inequality in (2.12) holds. Then $\mathcal{L}^{\hat{v},0}$ is critical and 1 is its ground state. On the other hand, thanks to Assumption 2.8 and $\hat{v} \in C^2(E)$, the adjoint operator $(\mathcal{L}^{\hat{v},0})^*$ exists and its ground state is $\hat{v}^2 m(y)$; see Theorem 5.1.10 in Pinsky (1995).

Now let $\hat{p}^0(s,y,z)$ be the density associated to $\mathcal{L}^{\hat{v},0}$; see (2.4) in Pinchover (2004) for its construction. It follows that

$$h^T(t,y) = \int_E \hat{p}^0(T-t,y,z)\frac{1}{\hat{v}(z)}\,dz.$$

Then the statement follows from Corollary 5.2 in Pinchover (2004). In this corollary, we choose $k_P^M = \hat{p}^0$, $f = (\hat{v})^{-1}$, $\varphi = 1$, and $\varphi^* = \hat{v}^2 m(y)$. Then this corollary is applicable given the second integral inequality in (2.12) holds, which ensures the numerator K is finite. That $K > 0$ follows because both $\hat{v}$ and m are strictly positive, and $\int_\alpha^\beta \hat{v}^2\, m(y)\,dy < \infty$. □

With these results, Lemma 2.15 and Theorems 2.16 and 2.17 are now proved.

Proof of Lemma 2.15. The proof of Proposition 5.5 show that for all $y \in E$ D^T and $\hat{D}$ are $(\mathbb{P}^y, (\mathcal{F}_t)_{t\geq 0})$ martingales and that $\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}|_{\mathcal{F}_t} = D^T_t$, $\frac{d\hat{\mathbb{P}}^y}{d\mathbb{P}^y}|_{\mathcal{F}_t} = \hat{D}_t$. Recalling (5.29), the limit in (2.21) holds provided that:

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty} \frac{h^T(t,Y_t)}{h^T(0,y)}\,\mathcal{E}\left(-\int a\Delta\frac{h^T_y}{h^T}\,d\hat{B}\right)_t = 1. \tag{5.31}$$

To prove this identity, it follows from Proposition 5.7 that, $\hat{\mathbb{P}}^y$-a.s., $\lim_{T\to\infty}\frac{h^T(t,Y_t)}{h^T(0,y)} = 1$. Recall from (5.24) that $\frac{h^T(t,Y_t)}{h^T(0,y)} = L^T_t$, where $L^T_t := \mathcal{E}(\int a\frac{h^T_y}{h^T}\,d\hat{W})_t$. Then $\{L^T_\cdot\}_{T\geq 0}$ is a sequence of positive $\hat{\mathbb{P}}^y$-local martingales such that $\hat{\mathbb{P}}^y$-$\lim_{T\to\infty} L^T_t = 1$. As a result, it follows from Fatou's lemma that $1 \geq \lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}[L^T_t] \geq \mathbb{E}^{\hat{\mathbb{P}}^y}[\liminf_{T\to\infty} L^T_t] = 1$, which implies $\lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}\left[|L^T_t - 1|\right] = 0$

due to Scheffé's lemma. Now Lemma 4.13 yields $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\left[\int a\frac{h^T_y}{h^T}\,d\hat{W}, \int a\frac{h^T_y}{h^T}\,d\hat{W}\right]_t = 0$, where $\int a\frac{h^T_y}{h^T}\,d\hat{W}$ is the stochastic logarithm of L^T. Observing that $\|\Delta\|^2$ is a constant (due to Assumption 2.11), the previous identity implies that $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\left[\int a\Delta\frac{h^T_y}{h^T}\,d\hat{B}, \int a\Delta\frac{h^T_y}{h^T}\,d\hat{B}\right]_t = 0$. This gives $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\int_0^t a\Delta\frac{h^T_y}{h^T}\,d\hat{B} = 0$, which implies $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\mathcal{E}\left(\int a\Delta\frac{h^T_y}{h^T}\,d\hat{B}\right)_t = 1$, i.e., the second term on the left-hand-side of (5.31) also converges to 1. This concludes the proof of (5.31). □

5.3. *Proof of main results in Section 2.2.*

Proof of Theorem 2.16. Let $\mathcal{S}_T$ be either $\{\sup_{u\in[0,t]}\left|r^T_u - 1\right| \geq \epsilon\}$ or $\{\left[\Pi^T, \Pi^T\right]_t \geq \epsilon\}$; both of them are $\mathcal{F}_t$-measurable. It follows from Theorem 2.5 and Remark 2.6 part iii) that

$$\lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}\left[\left.\frac{d\mathbb{P}^{T,y}}{d\hat{\mathbb{P}}^y}\right|_{\mathcal{F}_t} 1_{\mathcal{S}_T}\right] = 0. \tag{5.32}$$

On the other hand, (2.21) and Scheffé's lemma combined imply that $\lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}\left[\left|\left.\frac{d\mathbb{P}^{T,y}}{d\hat{\mathbb{P}}^y}\right|_{\mathcal{F}_t} - 1\right|\right] = 0$. Hence, combining the previous identity with (5.32), it follows that $\lim_{T\to\infty}\hat{\mathbb{P}}^y(\mathcal{S}_T) = 0$. Now, recall that $\hat{\mathbb{P}}^y \sim \mathbb{P}$ on $\mathcal{F}_t$ from Proposition 5.4, we conclude that

$$\lim_{T\to\infty}\mathbb{P}^y(\mathcal{S}_T) = \lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}\left[\left.\frac{d\mathbb{P}^y}{d\hat{\mathbb{P}}^y}\right|_{\mathcal{F}_t} 1_{\mathcal{S}_T}\right] = 0,$$

where the last equality follows from the dominated convergence theorem. □

Proof of Theorem 2.17. First, following a similar argument as that in the proof of Proposition 2.15 and using (5.30), we have $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}X^{0,T}_t/X^{\hat{\pi}}_t = 1$. On the other hand, Theorem 2.16 part a), combined with the equivalence between $\mathbb{P}$ and $\hat{\mathbb{P}}^y$, implies $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}X^{1,T}_t/X^{0,T}_t = 1$. Hence the last two identities combined gives $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\hat{r}^T_t = 1$. Now recall that $\hat{\pi}$ is the optimal portfolio for the logarithmic investor under $\hat{\mathbb{P}}^y$, it then follows from the numéraire property of $\hat{\pi}$ that $\hat{r}^T_\cdot$ is a $\hat{\mathbb{P}}^y$-supermartingale. This induces $\lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}\left[|\hat{r}^T_t - 1|\right] = 0$, thanks to Fatou's lemma and Scheffé's lemma. As a result, the statements follow from applying Lemma 4.13 under probability $\hat{\mathbb{P}}^y$ and switching back to the probability $\mathbb{P}^y$ at last. □

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