Advances in Mathematical Sciences and Applications
Vol. , No. (200x), pp.

Gakkōtosho
Tokyo, Japan

SOME PROBLEMS OF SHAPE OPTIMIZATION ARISING IN STATIONARY FLUID MOTION

Luigi C. Bersell
Dipartimento di Matematica Applicata "U.Dini", Università di Pisa
Via Bonanno 25/b, 56126 Pisa, Italy
(berselli@dma.unipi.it)

Paolo Guasoni
Dipartimento di Matematica "L.Tonelli", Università di Pisa
Via Buonarroti 2, 56127 Pisa, ITALY
Department of Mathematics and Statistics, Boston University
111 Cummington St, Boston, MA 02215, USA
(guasoni@dm.unipi.it)

Abstract. We investigate the existence of a drag-minimizing shape for two classes of optimal-design problem of fluid mechanics, namely the *vector Burgers* equations and the *Navier-Stokes equations.* It is known that the two-dimensional Navier-Stokes problem of shape optimization has a solution in any class of domains with at most l holes. We show, for the Burgers equation in three dimensions, that the existence of a minimizer still holds in the classes $\mathcal{O}_{c,r}(C)$ and $\mathcal{W}_w(C)$ introduced by Bucur and Zolésio. These classes are defined by means of capacitary constraints at the boundary. For the 3D Navier-Stokes equations we prove some results of existence of drag-minimizing shape, under additional assumptions on the class of domains to be considered. We also discuss how these assumptions critically depend on the definition of weak solutions for Navier-Stokes equations and, more specifically, on the characterization of the spaces in which it is possible to prove the uniqueness for the *linear Stokes* problem.

Received .
AMS Subject Classification: 35Q30, 49Q10, 49J10
Keywords and phrases: optimal shape, stationary fluid motion.

1 Introduction

In this paper we study the so called "submarine problem" and related problems: suppose that a body K, with given volume, is moving with velocity independent of time in an incompressible viscous fluid. It is well-known that the velocity $u(x)$ of the fluid, with respect to the body, can be found as the solution to the stationary Navier-Stokes equations:

$$\begin{cases} -\nu\Delta u_\Omega + (u_\Omega \cdot \nabla)u_\Omega + \nabla p = f \\ \nabla \cdot u_\Omega = 0 \end{cases} \qquad \text{in } \Omega \tag{NS}$$

with boundary conditions

$$\begin{cases} u_\Omega = 0 & \text{on} \quad \partial K, \\ \\ u_\Omega = u^\infty & \text{on} \quad \partial C, \end{cases} \tag{1}$$

where u^∞ is a constant vector. Here $C \subset \mathbf{R}^3$ is an open set, $K \subset C$ is a compact set, $\Omega = C \setminus K$, and, for simplicity, we assume that C contains the origin of $\mathbf{R}^3$ and that it is contained in a cube of side $2a$. The term f represents an external force, as gravity for example. Recall that the vector $u : \Omega \to \mathbf{R}^3$ is the velocity, the scalar $p : \Omega \to \mathbf{R}$ is the pressure, while the constant $\nu > 0$ is the kinematic viscosity.

Provided that (NS) admits a unique weak solution $u_\Omega \in H^1(\Omega; \mathbf{R}^3)$, (in the sequel we use the customary vector-valued Sobolev spaces, see Adams [1]) the resistance of K can be written in terms of u_Ω (see Section 3.1), so we can define a shape-functional $\Omega \longrightarrow J(u_\Omega)$. The first issue addressed in this paper is the following: under which assumptions does an optimal shape exists?

A standard approach to prove existence of minima is the "Direct Method of the Calculus of Variations": if we find a topology (or a convergence) such that:

i) the class of domains is a compact set;

ii) the functional is lower semicontinuous;

then, we have a minimum.

In our case the resistance functional J turns out to be a lower semicontinuous functional with respect to the weak convergence in the Sobolev space $H^1(\Omega; \mathbf{R}^3)$. This means that the above condition ii) is satisfied if we choose a convergence such that: $\Omega_j \to \Omega$ implies $u_{\Omega_j} \rightharpoonup u_\Omega$. Indeed, it can be shown that if solutions converge weakly in $H^1(\Omega; \mathbf{R}^3)$, then they also converge strongly.

In dimension two, the optimal shape problem for the Navier-Stokes equations has been solved by Šverák [16], who proved the existence of an optimum in the class $\mathcal{O}_l(C)$, defined as follows:

$$\mathcal{O}_l(C) = \{\Omega \subset C : \#(C \setminus \Omega) \leq l\},$$

where $\#$ denotes the number of connected components, and l is a positive integer.

The class $\mathcal{O}_l(C)$ is endowed with the complementary Hausdorff topology, that guarantees the class itself to be compact. Then in [16] it is proved that Hausdorff convergence of two-dimensional domains in $\mathcal{O}_l(C)$ implies strong convergence of the solutions both of (NS) and of the elliptic equation $-\Delta u_\Omega = f$ with Dirichlet boundary data.

In higher dimensions, the results are substantially different. Indeed, Šverák himself observed that there are sequences of domains in $\mathcal{O}_l(C)$, converging in the Hausdorff topology, for which the corresponding solutions do not converge to the solution of the problem in the limiting domain (for an example, see Cioranescu and Murat [5]). The reason is, roughly speaking, the following: in two dimensions the connected components condition enforces a upper bound on capacity. On the contrary, this link vanishes in three or more dimensions.

In this spirit, Bucur and Zolésio [3, 4] exhibited other classes of n-dimensional domains satisfying the above conditions i) and ii) with respect to the Hausdorff topology, for a broad range of functionals and state equations. In reference [3] Bucur and Zolésio introduced the class $\mathcal{O}_{c,r}(C)$ (for the precise definition, see Section 1.2). Using techniques coming from potential theory, they proved that $\mathcal{O}_{c,r}(C)$ is closed in the Hausdorff topology, and that $\Omega_j \xrightarrow{H^c} \Omega$ implies $u_{\Omega_j} \to u_\Omega$ in the strong topology of $\mathrm{H}^1_0(C)$. This guarantees the convergence of solutions for the equation $-\Delta u_\Omega = f$, and therefore the existence of extrema for several shape functionals.

In [4] Bucur and Zolésio weakened the capacitary constraint in the definition of $\mathcal{O}_{c,r}(C)$, and generalized their previous results to a larger class of domains, $\mathcal{W}_w(C)$, which is defined by a Wiener-type condition at the boundary.

In this paper, we discuss whether in three dimensions the problem (NS) admits a resistance-minimizing shape in the classes $\mathcal{O}_{c,r}(C)$ and $\mathcal{W}_w(C)$. In Section 2, we start by studying the nonlinear elliptic problem of the vector Burgers equations, for which we prove the existence of an optimum, as in the case of the Laplace equation. In Section 3 we investigate the problem for the Navier-Stokes equations. It turns out that an important role is played by the definition of weak solution for (NS). Indeed, (at least) two different definitions are available: we show the existence of an optimal shape for classes of domains where these two definitions coincide. In its full generality, the problem remains open, and it seems to be strictly connected to a long-dated uniqueness question in fluid-dynamics, discussed in Section 4.

1.1 Hausdorff convergence

In this section we recall a few basic facts on Hausdorff convergence that we shall use in the sequel.

Definition 1.1. *On the class $\mathcal{O} = \{\Omega \text{ open set} : \ \Omega \subset C\}$, where C is a bounded set, we define the Hausdorff complementary metric*

$$\rho(\Omega_1, \Omega_2) = \sup_{x \in \overline{C}} |d_{\Omega_1}(x) - d_{\Omega_2}(x)| \qquad \forall\, \Omega_1, \Omega_2 \in \mathcal{O},$$

where $d_\Omega(x) = dist(x, \overline{C} \setminus \Omega)$.

Let Ω_n, Ω be open sets. We denote by $\Omega_n \xrightarrow{H^c} \Omega$ the convergence in the Hausdorff complementary topology.

Proposition 1.2. *For a given $A \in \mathcal{O}$, we denote by A^ε the following set $A^\varepsilon = \{x \in A : dist(x, \overline{C} \setminus A) > \varepsilon\}$. Then, the following conditions are equivalent:*

a) $\Omega_n \xrightarrow{H^c} \Omega$;

b) $\forall\, \varepsilon > 0\ \exists\, n_\varepsilon$ *such that* $\Omega_n \supset \Omega^\varepsilon$, $\forall\, n \geq n_\varepsilon$; *furthermore,* $\forall\, \varepsilon > 0\ \exists\, n_\varepsilon$ *such that* $\Omega_n^\varepsilon \subset \Omega$, $\forall\, n \geq n_\varepsilon$.

Remark 1.3. *Condition b) implies that if $\Omega_n \xrightarrow{H^c} \Omega$ then, for each compact K, it eventually holds $\Omega_n \supset K$.*

An important property of the Hausdorff convergence is the following.

Proposition 1.4. *Let C be a bounded set. Then the class of open sets $\mathcal{O}$ is compact with respect to the Hausdorff convergence.*

1.2 Compactness results for $\mathcal{O}_{c,r}(C)$ and $\mathcal{W}_w(C)$

In this section we recall the main results of [3, 4]. In particular we give some definitions and the basic compactness results used to prove the existence of the shape minimizer.

Definition 1.5. *An open set Ω has the* (r, c)-capacity density condition *if:*

$$\forall\, x \in \partial\Omega \qquad \frac{\mathcal{C}(\Omega^c \cap B_r(x), B_{2r}(x))}{\mathcal{C}(B_r(x), B_{2r}(x))} \geq c,$$

where $B_r(x)$ is the ball (open or closed is equivalent, see [3]) centered at x and with radius r. Moreover $\mathcal{C}(A, B)$ denotes the capacity of A with respect to B. We define the following spaces:

$$\mathcal{O}_{c,r}(C) = \{\Omega \subset C \ : \forall\, r_0,\ 0 < r_0 < r,\ \Omega \textit{ has the } (r_0, c)\textit{-capacity density condition}\}$$

where $r < 1$, and

$$\mathcal{W}_w(C) = \{\Omega \subset C \ : \forall\, x \in \partial\Omega\ \forall\, r, R,\ 0 < r < R < 1,$$
$$\int_r^R \left(\frac{C(\Omega^c \cap B_r(x), B_{2r}(x))}{C(B_r(x), B_{2r}(x))} \right) \frac{dt}{t} \geq w(r, R, x)\}$$

where $w : (0,1) \times (0,1) \times C \to \mathbf{R}^+$ is such that:

1) w *is lower semicontinuous in* x;

2) $\lim_{r \to 0} w(r, R, x) = \infty$, *locally uniformly on* x.

Remark 1.6. *If $\Omega \in \mathcal{O}_{c,r}(C)$ then we have $\Omega \in \mathcal{W}_w(C)$. In fact, this can be seen by choosing $w(r', R, x) = K(r', R) - c \log r'$, where K is a continuous function, increasing in R, and decreasing in r'. This means that $\mathcal{O}_{c,r}(C)$ is, in fact, a particular subclass of $\mathcal{W}_w(C)$.*

The following results are proved in [3, 4]:

Theorem 1.7. *Let $\Omega_n \in \mathcal{O}_{c,r}(C)$ (respectively $\mathcal{W}_w(C)$) be a sequence of open sets, such that $\Omega_n \xrightarrow{H^c} \Omega$. Then $\Omega \in \mathcal{O}_{c,r}(C)$ (respectively $\mathcal{W}_w(C)$).*

2 The Burgers equation

In this section we consider the vector Burgers equations, which are simple, prototype equations. Basically, they are the Navier-Stokes equations without the complication of a pressure gradient and of the isochoricity constraint. The one dimensional Burgers equation was initially proposed as a turbulence model; the multi-dimensional case was studied (in the non-stationary case) among the others by Kiselev and Ladyžhenskaya [9] in their fundamental paper on the existence and uniqueness of solutions for incompressible fluids, showing the difference between nonlinear parabolic systems based on the Laplace operator versus those based on the Stokes operator. For a survey of results and references on Burgers equations, see Heywood [8].

2.1 Existence and Uniqueness Theorem

In this section we show an existence and uniqueness result for the Burgers equations. To the authors knowledge, in the literature there are no such a kind of results for weak solution to the stationary Burgers (B) equations, especially if we consider the problem with nonzero boundary data. Hence, Theorem 2.2 below may be interesting by itself. We remark that the existence of solutions, for any viscosity (and in arbitrary domains), remains an open problem.

We consider the vector Burger equation, namely the following system of partial differential equations, for the unknown $u_\Omega : \ \Omega \to \mathbf{R}^3$.

$$-\nu \Delta u_\Omega + (u_\Omega \cdot \nabla)\, u_\Omega = f, \tag{B}$$

again with the boundary conditions (1) and we consider it as the trace on $\Omega := C \backslash K$ of g which is a smooth vector valued function, belonging at least to $H^1 := H^1(\Omega; \mathbf{R}^3)$.

We introduce the variational formulation of the Burgers equations, where $H^1_0 = H^1_0(\Omega; \mathbf{R}^3)$,

$$\text{find } w \in H^1_0 : \quad a(w,v) = (f,v) - a(g,v) \qquad \forall\, v \in H^1_0, \tag{2}$$

$$a(w,v) := \nu \int_\Omega \nabla w \nabla v\, dx + \int_\Omega (w \cdot \nabla)\, w\, v\, dx,$$

and the dependence on Ω of w is not written explicitly.

We recall that

$$\int_\Omega \nabla w \nabla v\, dx = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial w_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx,$$

while

$$\int_\Omega (w \cdot \nabla)\, w\, v\, dx = \sum_{i,j=1}^{3} \int_\Omega w_j \frac{\partial w_i}{\partial x_j} v_i\, dx.$$

The solution u_Ω of (B) is finally defined by $u_\Omega = w + g$.

To construct a solution we introduce, given a function $W \in H^1_0$, the following bilinear form

$$a_W(w,v) := \nu \int_\Omega \nabla w \nabla v\, dx + \int_\Omega (W \cdot \nabla)\, w\, v\, dx.$$

Let us show that, for W small enough, it is coercive in H_0^1 (its continuity is straightforward). Let us consider the following expression

$$a_W(w,w) = \nu\|\nabla w\|^2 + \int_\Omega (W\cdot\nabla)\, w\, w\, dx = \nu\|\nabla w\|^2 - \frac{1}{2}\int_\Omega \operatorname{div} W|w|^2\, dx,$$

where $\|\,.\,\|$ denotes the $(L^2(\Omega))^3$ norm.

We now recall the so-called *Ladyžhenskaya estimate*

$$\|u\|_{L^4(\Omega)} \le \sqrt{2}\|u\|^{1/4}\|\nabla u\|^{3/4} \qquad \forall\, u \in H_0^1(\Omega), \tag{3}$$

that is independent of Ω. This estimate, which is a particular case of some Gagliardo-Nirenberg inequalities, is of fundamental importance in the mathematical theory of incompressible viscous fluids (see Lemma 1 in the classical Ladyžhenskaya's book [10]). We also recall the Poincaré estimate, that implies

$$\|u\|^2 \le 4a^2\|\nabla u\|^2 \qquad \forall\, u \in H_0^1(\Omega), \tag{4}$$

for any open set Ω contained in a cube of side $2a$, see Nečas [15].

By using the above two inequalities we have

$$\begin{aligned}\left|\int_\Omega \operatorname{div} W|w|^2\, dx\right| &\le \|\nabla W\|\,\|w\|_{L^4}^2 \le 2\|\nabla W\|\,\|w\|^{1/2}\|\nabla w\|^{3/2}\\ &\le 2(2a)^{1/2}\|\nabla W\|\,\|\nabla w\|^2.\end{aligned}$$

Finally, we obtain that

$$a_W(w,w) \ge \frac{\nu}{2}\|\nabla w\|^2,$$

provided that

$$\|\nabla W\| \le \frac{\nu}{4(2a)^{1/2}}. \tag{5}$$

We can now apply the classical Lax-Milgram lemma to state the following result.

Lemma 2.1. *Let be given $W \in H_0^1$ that satisfies* (5)*. Then, there exists a unique solution to the variational problem*

$$\textit{find } w \in H_0^1: \quad a_W(w,v) = (f,v) - a(g,v) \qquad \forall\, v \in H_0^1.$$

Let us try to find an *a priori* estimate on the solution w. We have

$$\begin{aligned}\frac{\nu}{2}\|\nabla w\|^2 \le |a_W(w,w)| \;&=\; |(f,w) - a(g,w)| \le\\ &\le\; \|f\|\,\|w\| + \nu\|\nabla g\|\,\|\nabla w\| + \|g\|_{L^4}\|\nabla g\|\,\|w\|_{L^4}\\ &\le\; 2a\|f\|\,\|\nabla w\| + \nu\|\nabla g\|\,\|\nabla w\| + 2(2a)^{1/2}\|\nabla g\|^2\,\|\nabla w\|\end{aligned}$$

Finally, by using several times the Young inequality we have:

$$\frac{\nu}{4}\|\nabla w\|^2 \leq 3\nu\|\nabla g\|^2 + \frac{24a}{\nu}\|\nabla g\|^4 + \frac{12a^2}{\nu}\|f\|^2$$

and hence

$$\|\nabla w\|^2 \leq \frac{4}{\nu}\left(3\nu\|\nabla g\|^2 + \frac{24a}{\nu}\|\nabla g\|^4 + \frac{12a^2}{\nu}\|f\|^2\right) \stackrel{def}{=} \Phi(\nu, a, g, f). \tag{6}$$

Then, if

$$\frac{4}{\nu}\left(3\nu\|\nabla g\|^2 + \frac{48a^2}{\nu}\|\nabla g\|^4 + \frac{6a}{\nu}\|f\|^2\right) \leq \frac{\nu^2}{32a}$$

the map $T : W \to w$ maps a ball of H_0^1 into itself.

Let us show that, under suitable hypotheses the map T is a contraction. Consequently, its fixed point will be the unique solution of (2). We consider w_1, w_2 solution of the linear problem corresponding to the bilinear forms $a_{W_1}(w, v)$ and $a_{W_2}(w, v)$, respectively, We subtract the equation satisfied by w_2 to that one satisfied by w_1, we multiply it by $w_1 - w_2$, and we integrate over Ω to obtain

$$\nu\|\nabla(w_1 - w_2)\|^2 = \int_\Omega [(W_1 \cdot \nabla)\, w_1 - (W_2 \cdot \nabla)\, w_2]\,(w_1 - w_2)\, dx =$$

$$= \int_\Omega W_1 \nabla \frac{(w_1 - w_2)^2}{2}\, dx + \int_\Omega ((W_1 - W_2) \cdot \nabla)\, w_2 (w_1 - w_2)\, dx.$$

The first integral on the right hand side can be estimated (recall (3)-(4)) as follows

$$\left|\int_\Omega W_1 \cdot \nabla \frac{(w_1 - w_2)^2}{2}\, dx\right| \leq \frac{1}{2}\int_\Omega |\nabla \cdot W_1|\, |w_1 - w_2|^2\, dx$$

$$\leq (2a)^{1/2}\|\nabla W_1\|\, \|\nabla(w_1 - w_2)\|^2,$$

while the second

$$\left|\int_\Omega ((W_1 - W_2) \cdot \nabla)\, w_2 (w_1 - w_2)\, dx\right| \leq$$

$$\leq 2(2a)^{1/2}\|\nabla(W_1 - W_2)\|\, \|\nabla w_2\|\, \|\nabla(w_1 - w_2)\|.$$

We have then

$$\frac{\nu}{2}\|\nabla(w_1 - w_2)\|^2 \leq 2(2a)^{1/2}\|\nabla(W_1 - W_2)\|\, \|\nabla w_2\|\, \|\nabla(w_1 - w_2)\|$$

and by recalling (6) we infer that there exists $\nu_0(g, f, a)$ such that for $\nu > \nu_0$ the map T is a strict contraction, the constant ν_0 being defined by the relation

$$\frac{4(2a)^{1/2}}{\nu_0}\,\Phi(\nu_0, a, g, f) = 1. \tag{7}$$

We have finally proved the following result.

Theorem 2.2. *Let be given $\nu > \nu_0$, where ν_0 is defined by condition* (7)*, then the Burgers equations* (B) *admit a unique solution $u_\Omega \in H^1$.*

Remark 2.3. Theorem 2.2 shows the existence and uniqueness of solutions in a particular bounded set of H^1. The bound depends on the data ν, f, g. Then, it is an existence and uniqueness result for "small" data. Probably, by using different techniques, it is possible to prove an existence (but very unlikely a uniqueness) result for arbitrary ν. We preferred to use the above tokens since in the sequel we will need the uniqueness and we obtained both results just with elementary techniques.

2.2 The drag

In problem (B), the object K moves in under the action of an external field f. Therefore, the flow is going to be stationary only if a force $G(K, u_K)$ balances the friction and the field f. In other words, we set:

$$G(K, u_\Omega) = -\nu \int_{\partial K} \frac{\partial u_\Omega}{\partial \hat{n}} \, ds - \int_K f \, dx,$$

where $\hat{n}$ is the inward normal to ∂K. Note that the functional depends only on K, since $\Omega = C \backslash K$ and the solution u_Ω is determined uniquely by the data of the problem.

We set

$$\hat{u}^\infty = \begin{cases} 0 & \text{in} \quad C \\ u^\infty & \text{in} \quad \mathbf{R}^3 \backslash C \end{cases}$$

and we define $U^\infty = \phi \, \hat{u}^\infty$, where $\phi \in C_0^\infty(\mathbf{R}^3)$, with $0 \le \phi \le 1$ and

$$\phi = \begin{cases} 1 & \text{in} \quad B_b(0) \\ 0 & \text{in} \quad \mathbf{R}^3 \backslash B_{2b}(0) \end{cases}$$

for a sufficiently big $b > 0$, such that $\Omega \subset B_b(0)$ (the ball of radius b, centered in the origin). In this way $u_\Omega - U^\infty \in H_0^1$.

If we multiply (B) by $u_\Omega - U^\infty$ and we integrate over Ω we obtain that

$$-\nu \int_\Omega \Delta u_\Omega \, (u_\Omega - U_\infty) \, dx = \nu \int_\Omega |\nabla u_\Omega|^2 \, dx + \nu \int_{\partial K} \frac{\partial u_\Omega}{\partial \hat{n}} u^\infty \, ds, \tag{8}$$

where $\hat{n}$ is the outward unit vector. Furthermore,

$$\int_\Omega (u_\Omega \cdot \nabla) \, u_\Omega \, (u_\Omega - U_\infty) \, dx = \frac{1}{2} \int_\Omega u_\Omega \cdot \nabla |u_\Omega - U_\infty|^2 \, dx$$

$$= -\frac{1}{2} \int_\Omega (\nabla \cdot u_\Omega) \, |u_\Omega - U_\infty|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} u_\Omega \cdot \hat{n} \, |u_\Omega - U_\infty|^2 \, ds$$

and, since u_Ω vanishes on ∂K and equals u^∞ on ∂C, we have that

$$\int_\Omega (u_\Omega \cdot \nabla)\, u_\Omega\, (u_\Omega - U^\infty)\, dx = -\frac{1}{2}\int_\Omega (\nabla \cdot u_\Omega)|u_\Omega|^2\, dx. \tag{9}$$

Finally, by adding up these expressions we obtain

$$-\nu \int_{\partial K} \frac{\partial u_\Omega}{\partial \hat{n}} u^\infty\, dx = \nu \int_\Omega |\nabla u_\Omega|^2 - \frac{1}{2}\int_\Omega (\nabla \cdot u_\Omega)\, |u_\Omega|^2\, dx - \int_\Omega f u_\Omega\, dx.$$

The power is then given by the functional $F(K) = F(K, u_\Omega)$:

$$\begin{aligned} F(K) &= G(K)u^\infty = \\ &= \nu \int_\Omega |\nabla u_\Omega|^2\, dx - \frac{1}{2}\int_\Omega (\nabla \cdot u_\Omega)|u_\Omega|^2\, dx - \int_\Omega f u_\Omega\, dx - \int_K f u^\infty\, dx. \end{aligned} \tag{10}$$

2.3 Existence of an optimal shape

We are now ready to prove the existence of a resistance-minimizing shape in the classes $\mathcal{O}_{c,r}(C)$ and $\mathcal{W}_w(C)$.

Lemma 2.4. *Let $\{K_h\} \subset C$ be a sequence of compact sets such that $\Omega_h = C \setminus K_h \in \mathcal{W}_w(C)$, $\Omega_h \xrightarrow{H^c} \Omega$, and let $u_{\Omega_h} \in H^1$ be solutions of* (B) *in Ω_h. Then we have $u_{\Omega_h} \rightharpoonup u_\Omega$, where u_Ω is a H^1 solution in Ω of* (B)*.*

Proof. Since the sequence $\{u_{\Omega_h}\}$ is uniformly bounded in the $\mathrm{H}^1_0(C)$ norm, up to a subsequence, $\{u_{\Omega_h}\}$ weakly converges to some function u. We need to see that u is a weak solution of (B) on Ω. By assumption we have:

$$\int_C \nu \nabla u_{\Omega_h} \nabla \varphi\, dx + \int_C (u_{\Omega_h} \cdot \nabla) u_{\Omega_h} \varphi\, dx - \int_C f\varphi\, dx = 0 \qquad \forall\, \varphi \in H^1_0(C) \tag{11}$$

The weak convergence of u_{Ω_h} immediately implies that the first term passes to the limit. To see that also the second does, we write explicitly

$$\begin{aligned} &\int_C [(u_{\Omega_h} \cdot \nabla)u_{\Omega_h} - (u \cdot \nabla)u]\, \varphi\, dx = \\ &\quad = \int_C [(u_{\Omega_h} \cdot \nabla)u_{\Omega_h} - u_{\Omega_h} \nabla u]\, \varphi\, dx + \int_C [u_{\Omega_h} \nabla u - (v \cdot \nabla) v]\, \varphi\, dx = \\ &\quad\quad = \int_C u_{\Omega_h} \nabla (u_{\Omega_h} - u)\, \varphi\, dx + \int_C (u_{\Omega_h} - u) \nabla u\, \varphi\, dx \end{aligned}$$

and observe that the last two terms vanish as $h \to \infty$, thanks to the compactness of the embedding $H^1_0 \subset L^p$, for $2 \le p < 6$. This implies that $u = u_\Omega$ and satisfies equation (11). $\square$

The main existence theorem is stated as follows.

Theorem 2.5. *Let C be an open set, and $B \subset C$ a bounded open subset. Let $f \in L^2(C; \mathbf{R}^3)$, and ν as in Theorem 2.2. Then, for every $\gamma > 0$ the functional $F(K)$, defined in* (10)*, has at least one minimizer in the class:*

$$\mathcal{K}_{\gamma,w} = \{K \subset B : meas(K) \geq \gamma,\ C \setminus K \in \mathcal{W}_w(C)\}.$$

Proof. Let $K_h \in \mathcal{K}_{\gamma,w}$ be a minimizing sequence for F. We set $\Omega_h = C \setminus K_h$ and denote by u_{Ω_h} the solution to (B) in Ω_h such that $F(K_h) = F(K_h, u_{\Omega_h})$. Since $\{\Omega_h\}$ is a bounded sequence of open sets, we can extract a subsequence which converges to some set Ω in the Hausdorff complementary topology. By Theorem 1.7 we have that $\Omega \in \mathcal{W}_w(C)$ and, by Lemma 2.4, u_{Ω_h} converges weakly to u_Ω, a solution to (B) in Ω.

Since F is weakly lower semicontinuous, we have:

$$\liminf_{h\to\infty} F(K_h, u_{\Omega_h}) \geq F(K, u_\Omega)$$

and the proof is complete. □

3 The Navier-Stokes Equations

In this section we consider the real problem related to the Navier-Stokes equations (NS), with the boundary conditions (1). By recalling the Stokes paradox, if $n = 2$ and C is unbounded, then a solution exists only if u^∞ is identically zero (see for instance Ladyžhenskaya [10] or Galdi [6]). This means that in two dimensions it is meaningless to take $C = \mathbf{R}^2$, with the condition $u_\Omega \to u^\infty$ at infinity. In higher dimension this is no longer the case, and C may represent the whole space as well.

Definition 3.1. *We define the following classes of divergence-free functions, following the notation of Temam [17]:*

$$\begin{aligned} \mathcal{V}(\Omega) =& \{u :\ u \in \mathcal{D}(\Omega) := (C_0^\infty(\Omega))^3,\ \nabla \cdot u = 0\} \\ V(\Omega) =& \overline{\mathcal{V}}(\Omega) \ \textit{in the strong topology of } H_0^1(\Omega; \mathbf{R}^3) \\ V^\circ(\Omega) =& \{u :\ u \in H_0^1(\Omega; \mathbf{R}^3),\ \nabla \cdot u = 0\} \end{aligned}$$

It is immediately seen that $V(\Omega) \subseteq V^\circ(\Omega)$. The reverse inclusion does not hold for an arbitrary open set. For a class of unbounded domains, as that ones with *exits at infinity* (for example channels with increasing cross section or the aperture domain), it is known that $V(\Omega) \not\subset V^\circ(\Omega)$, and examples of arbitrary finite co-dimension are known (in fact, in such examples the co-dimension is equal to the number of exits, see Ladyžhenskaya and Solonnikov [11]). For the question if $V(\Omega) = V^\circ(\Omega)$, in bounded (non-regular) sets, see the discussion in Section 4.

Now we can define the notion of weak solutions for problem (NS).

Definition 3.2. *Let $w \in C^\infty(C, \mathbf{R}^3)$ be a function such that $w = 0$ in a neighborhood of K, $w = u^\infty$ in a neighborhood of ∂C and $\nabla \cdot w = 0$ in C.*
*We say that a function $u \in H^1(C; \mathbf{R}^3)$ is a V-*solution *for problem* (NS) *if:*

i) $u - w \in V(\Omega)$;

ii) $\displaystyle\int_C [\nu \nabla u \ \nabla\varphi + ((u \cdot \nabla)u - f)\varphi] \, dx = 0 \qquad \forall\, \varphi \in V(\Omega).$

If we replace $V(\Omega)$ with $V^\circ(\Omega)$ in conditions i) and ii), we have a V°-solution.

Remark 3.3. *This definition is clearly independent of the choice of w.*

Remark 3.4. *In $\mathbf{R}^3$ we can choose $w = curl\ v\phi$, where v is the linear function $(x, y, z) \to (-u_z^\infty y + u_y^\infty z, -u_x^\infty z, 0)$, while $\phi \in C^\infty(C)$, $\phi = 0$ on K, and $\phi = 1$ on ∂C.*

It is immediate that a V-solution is also a V°-solution, therefore existence theorems only need to be proved for the former type. On the other hand, V-uniqueness generally does not imply V°-uniqueness.

The existence and uniqueness of weak solutions for (NS), regardless of the regularity of the boundary $\partial\Omega$, follow from Lions [13] and Murat and Simon [14].

Theorem 3.5. *The system* (NS) *admits at least one V-solution (u_Ω, p_Ω) in the space of distributions.*

Uniqueness is known up to dimension four, and only for V-solutions, and we need an additional lower bound on the viscosity ν:

Theorem 3.6. *Let $n \leq 4$. Then there exists $\nu_0(C, K, u^\infty)$ such that for $\nu \geq \nu_0$ the system* (NS) *has only one V-solution.*

3.1 The drag

In this section we derive the drag relative to the Navier-Stokes equations. In problem (NS), the object K moves in a viscous incompressible fluid under the action of an external field f. Again, the flow is going to be stationary only if a force $G(K) = G(K, u_\Omega, p_\Omega)$ balances the friction, the pressure, and the field f. In other words:

$$G(K, u_\Omega, p_\Omega) = -\int_{\partial K} \left(\nu \frac{\partial u_\Omega}{\partial \hat{n}} + p_\Omega \, \hat{n} \right) ds - \int_K f \, dx,$$

where $\hat{n}$ is the inward normal to ∂K.

The above formula can be substantially simplified using integration by parts, the Stokes formula, and the boundary conditions of problem (NS). In particular, it is possible to obtain an expression independent of the pressure p_Ω.

As we did above, we multiply the first equation in (NS) by $(u_\Omega - u^\infty)$ and integrate over $\Omega = C \setminus K$. The term concerning the Laplace operator is the same as (8). Regarding the nonlinear term, we obtain the same equation as in (9), but in this case the vector u_Ω is divergence-free, and we have:

$$\int_\Omega (u_\Omega \cdot \nabla) \, u_\Omega \, (u_\Omega - u^\infty) \, dx = 0.$$

Finally, the term with the pressure can be treated as follows

$$\int_\Omega \nabla p_\Omega \, (u_\Omega - u^\infty) \, dx = \int_{\partial\Omega} p_\Omega \, (u_\Omega - u^\infty) \cdot \hat{n} \, dx = - \int_{\partial K} p_\Omega \, u^\infty \cdot \hat{n} \, dx.$$

Summing up, it follows that

$$\nu \int_\Omega |\nabla u_\Omega|^2 \, dx + \nu \int_{\partial K} \frac{\partial u_\Omega}{\partial \hat{n}} u^\infty \, ds = - \int_{\partial K} p_\Omega u^\infty \cdot \hat{n} \, ds + \int_\Omega f(u_\Omega - u^\infty) \, dx.$$

This implies that the power is given by the functional $F(K)$:

$$F(K) = G(K) u^\infty = \nu \int_\Omega |\nabla u_\Omega|^2 \, dx - \int_\Omega f(u_\Omega - u^\infty) \, dx - u^\infty \int_K f \, dx. \tag{12}$$

Since V-solutions are unique, the functional $F(K)$ is well defined.

3.2 Existence of an optimal shape

We are now ready to prove the existence of a resistance-minimizing shape for the Navier-Stokes equations. Throughout this section, we need the following technical assumption:

Definition 3.7. *We define the following class $\mathcal{A}$ of opens subsets of C, by means of the following conditions:*

i) $\mathcal{A} \subset \mathcal{W}_w(C)$;

ii) $\mathcal{A}$ *is closed under Hausdorff complementary convergence;*

iii) for any $\Omega \in \mathcal{A}$, *we have that* $V(\Omega) = V^\circ(\Omega)$.

Remark 3.8. *An example of class $\mathcal{A}$ with the above properties is obtained by all bounded sets which satisfy a* uniform Lipschitz *condition (see for instance Bucur and Zolésio [3] and Adams [1]). For a discussion on more general classes of domains where iii) holds, see the next section.*

Lemma 3.9. *Let $\mathcal{A}$ as in Definition 3.7. Let $K_h \subset C$ be a sequence of compact sets such that $\Omega_h = C \setminus K_h \in \mathcal{A}$, $\Omega_h \xrightarrow{H^c} \Omega$, and let u_{Ω_h} be solutions of* (NS) *in Ω_h. Then we have $u_{\Omega_h} \rightharpoonup u_\Omega$, where u_Ω is a solution in Ω.*

Proof. By Remarks 1.3 and 3.3 we can choose the same w for all the weak solutions u_{Ω_h}. Since u_{Ω_h} is uniformly bounded in the $H^1(C)$ norm, it weakly converges up to a subsequence to some function u. Furthermore, $u - w \in \mathrm{H}^1_0(C)$.

We need to see that u is a weak solution of (NS) on Ω. For any $\varphi \in \mathcal{V}(\Omega)$, by Remark 1.3 we have that $\varphi \in \mathcal{V}(\Omega_h)$, eventually in h. For all such h, by assumption we have:

$$\int_C \nu \nabla u_{\Omega_h} \nabla \varphi \, dx + \int_C (u_{\Omega_h} \cdot \nabla) u_{\Omega_h} \, \varphi \, dx - \int_C f \varphi \, dx = 0$$

And this equation passes to the limit as in Lemma 2.4.

It remains to show that $u - w \in V^\circ(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$: again, we have that $\varphi \in C_c^\infty(\Omega_h)$, eventually in h. It follows that:

$$\int_\Omega (u_{\Omega_h} - w)\nabla\varphi\, dx = \int_{\Omega_h} (u_{\Omega_h} - w)\nabla\varphi\, dx = 0$$

Since $u_{\Omega_h} \rightharpoonup u$, we obtain $u \in V^\circ(\Omega) = V(\Omega)$, as needed. □

The existence theorem is then stated as follows:

Theorem 3.10. *Let C be an open set, and $B \subset C$ a bounded open subset. Let $f \in L^2(C;\mathbf{R}^3)$, and ν as in Theorem 2.2. Then for every $\gamma > 0$ the functional $F(K)$, defined in* (12)*, has at least one minimizer in the class:*

$$\mathcal{K}_{\gamma,w} = \{K \subset B : meas(K) \geq \gamma,\ C \setminus K \in \mathcal{A}\}.$$

Proof. Let $K_h \in \mathcal{K}_{\gamma,w}$ be a minimizing sequence for F. We set $\Omega_h = C \setminus K_h$ and denote by u_{Ω_h} the solution to (NS) in Ω_h. Since $\{\Omega_h\}$ is a bounded sequence of open sets, we can extract a subsequence which converges to some set Ω in the Hausdorff complementary topology. By Theorem 1.7 we have that $\Omega \in \mathcal{W}_w(C)$, and, by Lemma 3.9, u_{Ω_h} converges weakly to u_Ω, a solution to (NS) in Ω. Since F is weakly lower semicontinuous, it follows that u is a minimum. □

Remark 3.11. *In the above theorem, we added the boundedness condition $K \subset B$ to avoid minimizing sequences that do not converge. For example, if $C = \mathbf{R}^3$, one can imagine an unbounded minimizing sequence as follows: let D be a ball with the prescribed volume γ, and let K_h be equal to D, but stretched h times in the direction of u_∞, and $h^{-\frac{1}{n-1}}$ times in any orthogonal direction to u^∞. In this case K_h converges to a line parallel to u^∞, which has zero capacity and therefore is negligible as boundary condition. This means that the solution in the limiting domain is simply $u_\Omega = u^\infty$, which obviously leads to null resistance.*

4 Further remarks

In Theorem 3.10, we proved the existence of minima for classes of domains for which the equality $V(\Omega) = V^\circ(\Omega)$ holds. A natural question is whether this condition holds for all bounded sets Ω.

In fact, Šverák [16] proved such equality for any $\Omega \in \mathcal{O}_l(C)$ (with C a bounded subset of $\mathbf{R}^2$), and thus the existence of an optimal shape without the above restriction. Unfortunately, his proof relies crucially on stream functions, which are available only in two dimensions.

The above characterization problem has a long history: it was first clearly stated by Heywood [7], who recognized that the equality of $V(\Omega)$ and $V^\circ(\Omega)$ is equivalent to the uniqueness of the linear Stokes equations. He provided an example of an unbounded open set where the two spaces differ, as well as sufficient conditions which guarantee equality for bounded and exterior domains, with C^2 regularity. See the discussion in Galdi [6], § III.4.

After that, several authors improved Heywood's results. Observe that in Lions [12] it was proved (several years before Heywood paper) that $V(\Omega) = V^{\circ}(\Omega)$ for all bounded Lipschitz domains, even if this regularity of the boundary was not stated explicitly in the hypotheses. Later, Temam [17] recognized the necessity of this assumption. The known best result is given in reference [2], where Bogovskiĭ proved the equality $V(\Omega) = V^{\circ}(\Omega)$ for finite unions of sets which are star-shaped with respect to a ball. To our knowledge, this is the largest class where equality is known for $n \geq 3$. Still, it is not large enough to include $\mathcal{O}_{c,r}(C)$ or $\mathcal{W}_w(C)$.

On the other hand, no bounded counterexamples are known, and the result of Šverák in dimension two suggests that topology (or capacity constraints) may be a more important issue than regularity.

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, 1975, Pure and Applied Mathematics, Vol. 65.

[2] M.E. Bogovskiĭ, *Solutions of some problems of vector analysis, associated with the operators* div *and* grad, Theory of cubature formulas and the application of functional analysis to problems of mathematical physics, Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1980, pp. 5–40, 149.

[3] D. Bucur and J.-P. Zolésio, *N-dimensional shape optimization under capacitary constraint*, J. Differential Equations **123** (1995), no. 2, 504–522.

[4] D. Bucur and J.-P. Zolesio, *Wiener's criterion and shape continuity for the Dirichlet problem*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 4, 757–771.

[5] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980), Pitman, Boston, Mass., 1982, pp. 98–138, 389–390.

[6] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I,* Linearized steady problems, Springer-Verlag, New York, 1998.

[7] J.G. Heywood, *On uniqueness questions in the theory of viscous flow*, Acta Math. **136** (1976), no. 1-2, 61–102.

[8] ———, *Remarks on the possible global regularity of solutions of the three-dimensional Navier-Stokes equations*, Progress in theoretical and computational fluid mechanics (Paseky, 1993), Longman Sci. Tech., Harlow, 1994, pp. 1–32.

[9] A.A. Kiselev and O.A. Ladyženskaya, *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid*, Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 655–680.

[10] O.A. Ladyžhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1969, Second English edition, revised and enlarged. Mathematics and its Applications, Vol. 2.

[11] O.A. Ladyženskaya and V.A. Solonnikov, *Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier-Stokes equation*, Boundary value problems of mathematical physics and related problems 9, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), **59**, (1976), 81–116.

[12] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier–Villars, Paris, 1969.

[13] J.-L. Lions, *Some aspects of the optimal control of distributed parameter systems*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1972, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 6.

[14] F. Murat and J. Simon, *Sur le contrôle par un domaine géométrique*. Pré-publication du Laboratoire d'Analyse Numérique, no. 74015, Université de Paris 6 (1976), 222 pages.

[15] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[16] V. Šverák, *On optimal shape design*, J. Math. Pures Appl. (9) **72** (1993), no. 6, 537–551.

[17] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland Publishing Co., Amsterdam, 1977, Studies in Mathematics and its Applications, Vol. 2.