# UTILITY MAXIMIZATION WITH INFINITELY MANY ASSETS

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Abstract. We study the problem of utility maximization from terminal wealth in a semimartingale model with countably many assets.

After discussing in this context the appropriate notion of admissible strategy, we give a characterization result for the superreplication price of a contingent claim.

Utility maximization problems are then studied with the convex duality method, and we extend finite-dimensional results to this setting.

The existence of an optimizer is proved in a suitable class of *generalized* strategies: this class has also the property that maximal utility is the limit of maximal utilities in finite-dimensional submarkets.

Counterexamples are then given, which illustrate several phenomena which arise in presence of infinitely many assets.

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## 1. Introduction

The classical problem of utility maximization, which goes back to Arrow and Debreu, was first studied in continuous-time by Merton [23, 24] with a stochastic control approach.

The modern approach to this problem is based on the dual characterization of portfolio processes, a technique developed by a number of authors (see for example Karatzas et al. [14, 15] and the references therein), which is commonly referred to as the convex duality method. This approach allows to drop the assumption that asset prices are Markov processes, and extends the basic results of Arrow and Debreu, which dictate that marginal utility of optimal terminal wealth should be proportional to a state price density.

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The most general results in this area are due to Kramkov and Schachermayer [19, 20], and are valid for a general semimartingale models. Such a model consists in a market with $d$ risky assets and one riskless asset. To simplify notation, the latter is used as numéraire, and is assumed identically equal to 1. The prices of risky assets are modeled by a $d$-dimensional semimartingale $(S^i)_{i\le d}$, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$.

An economic agent endowed with initial capital $x$ invests in these assets, so as to maximize the expected utility from terminal wealth. This problem can be written as:

$$\max_{H\in\mathcal{A}} E\left[U\left(x+\int_0^T H_t dS_t\right)\right] \tag{1.1}$$

where $\mathcal{A}$ is the class of admissible strategies, that is the set of predictable, $S$-integrable processes $H$ such that the process $(\int HdS)_t$ is uniformly bounded from below.

The essential result for convex duality methods is the dual characterization of the superreplication price of a non-negative claim $X$ (see, for instance [6, 7]):

$$\sup_{Q\in\mathcal{M}_e} E_Q[X] = \inf\left\{x : X \le x + \int_0^T H_s dS_s \ \text{ for some } H\in\mathcal{A}\right\}, \tag{1.2}$$

where $\mathcal{M}_e$ is the set of equivalent martingale measures and the infimum in the right-hand side is in fact a minimum.

The above equality implies a polarity between the set of claims which are super-replicable at price 1 and the set of the densities of martingale measure: an exact bipolar relation holds if the set of martingale densities is replaced by its closed solid hull. This result is the main tool in the papers by Kramkov and Schachermayer [19, 20].

In the present paper, we study the problem of utility maximization in a financial market with countably many assets: the setting essentially corresponds, in the literature on Arbitrage Pricing Theory, to a continuous-time extension of the stationary market originally considered by Ross [26] (see also Huberman [10]). It can also be seen as a special case of the large financial market setting considered by Kabanov and Kramkov [12], which consists of a sequence of finite-dimensional markets, possibly defined on different probability spaces. We consider a sequence of semimartingales living on a fixed probability space, which represent the prices of risky assets, as in the model considered by Björk and Näslund [2].

Formulating the utility maximization problem in this setting requires three basic ingredients: *i*) a no-arbitrage assumption, to make optimization nontrivial, *ii*) a set of admissible strategies allowing investments in infinitely many assets and hence *iii*) a definition for the infinite-dimensional stochastic integral $\int_0^T H_t dS_t$. In addition, to employ the convex duality method, we need *iv*) a dual characterization of superreplicable claims, analogous to (1.2).

It is rather clear that these issues are tightly connected. In fact, they all concur to define the space of attainable claims, the object which ultimately determines the solution (and the solvability) of the problem.

A theory of stochastic integration with respect to a sequence of semimartingales is developed in [5]: we recall the definition of integrable process and refer to that paper for all basic results. With this definition, we can define several classes of admissible strategies which, in some sense, generalize the finite-dimensional definition of admissibility. The natural question is then to identify the class which is most appropriate for optimization problems.

To exclude arbitrage, we assume the existence of a martingale measure (in the sense of [19]), an assumption equivalent to the condition of No Free Lunch (NFL), introduced by Kreps [21]. It is important to notice that, unlike the finite-dimensional case, this condition is not relaxed to No Free Lunch with Vanishing Risk (NFLVR) (see Remark 2.2 for details).

The paper is organized as follows: in section 2 we describe our model in detail, define the various classes of admissible strategies, and discuss their mutual relationships. The main superreplication result is in section 3, which contains our answer to *iv*). Given a proper set of admissible strategies, we extend to infinite-dimensional markets the dual characterization of portfolio processes. This result paves the way to the convex duality approach to utility maximization problems, which is the treated in section 4. We prove that utility maximization over all finite-dimensional strategies is equivalent to maximizing utility over a suitable class of generalized strategies, and show the existence of an optimizer within this class.

The usual properties of the optimizer will then follow from the finite-dimensional semimartingale results of Kramkov and Schachermayer [19, 20].

## 2. The model

We consider the model of a financial market with countably many assets: we assume, as in [2], [4], that there is one fixed market which consist of a riskless asset $S^0$, used as numéraire, with price constantly equal to 1, and countably many risky assets, which are modeled by a sequence of semimartingales $(\mathbf{S}_t)_{t\in[0,T]} = ((S^i_t)_{t\in[0,T]})_{i=1}^{\infty}$, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, which satisfies the usual assumptions.

We begin our discussion with the following:

**Definition 2.1.**

i) A *$n$-elementary strategy* is a $\mathbb{R}^n$-valued, predictable process, integrable with respect to $(S^i)_{i\leq n}$. An *elementary strategy* is a strategy which is $n$-elementary for some $n$.

ii) Let $x \in \mathbb{R}^+$: an $n$-elementary strategy $H$ is said to be *x-admissible*, if $(H \cdot \mathbf{S})_t = \int_0^t \sum_{i \le n} H^i dS^i \ge -x$ a.s. An elementary strategy $H$ is called *admissible* if it is $x$-admissible for some $x \in \mathbb{R}_+$.

As usual the notation $H \cdot S$ denotes the stochastic integral process $\int H dS$: we point out that, though in Definition 2.1 ii), the semimartingale $\mathbf{S}$ is infinite-dimensional, the process $(H \cdot \mathbf{S})$ is just a standard stochastic integral in $\mathbb{R}^n$.

We denote by $\mathcal{H}^n$ the set of $n$-elementary admissible strategies and by $\mathcal{H}$ the set of admissible elementary strategies. In other words, elementary strategies are those involving only a finite number of assets. These strategies should be allowed by any reasonable definition of admissibility, therefore any no-arbitrage condition should exclude elementary arbitrage strategies. Since in finite-dimensional markets the absence of arbitrage is equivalent to the existence of (local) martingale measures, we obtain a set of necessary conditions for the absence of arbitrage.

More formally, following the notation of Delbaen and Schachermayer [6], we denote by $K_{0,n} = \{(H \cdot \mathbf{S})_T \mid H \in \mathcal{H}^n\}$ the linear space of claims attainable with $n$-elementary strategies starting from zero wealth, and by $C_{0,n} = K_{0,n} - L^0_+$ (resp. $C_n = C_{0,n} \cap L^\infty$) the convex cones of superreplicable (resp. bounded and superreplicable) claims. We set

$$K_0 = \bigcup_{n \ge 1} K_{0,n} \qquad C_0 = \bigcup_{n \ge 1} C_{0,n} \qquad C = \bigcup_{n \ge 1} C_n.$$

and define the following sets of (local) martingale measures:

$$\mathcal{M}^n = \{Q \ll P \mid (H \cdot \mathbf{S}) \text{ is a } Q\text{-local martingale for all } H \in \mathcal{H}^n\}$$
$$\mathcal{M}^n_e = \{Q \in \mathcal{M}^n \mid Q \sim P\}$$
$$\mathcal{M} = \bigcap_{n \ge 1} \mathcal{M}^n, \qquad \mathcal{M}_e = \bigcap_{n \ge 1} \mathcal{M}^n_e.$$

*Remark* 2.1. The above definition is given in analogy to [19], and it does not imply that each $S^i$ is a local martingale under any $Q \in \mathcal{M}_e$. In fact, the local martingale property is recovered for any $S^i$ that is locally bounded (see [6] fo details).

We recall the following

**Definition 2.2.** The process $\mathbf{S}$ satisfies the condition of

(EMM) *Equivalent Martingale Measure* if $\mathcal{M}_e \neq \emptyset$.

(NFL) *No Free Lunch* if $\overline{C}^* \cap L^\infty_+ = \{0\}$

(NFLVR) *No Free Lunch with Vanishing Risk* if $\overline{C} \cap L^\infty_+ = \{0\}$

where $\overline{C}^*$ and $\overline{C}$ denote the closure of $C$ respectively in the $\sigma(L^\infty, L^1)$ topology and in the $L^\infty$-norm.

The equivalence between conditions (EMM) and (NFL) follows from the so-called Kreps-Yan Theorem (see for instance [6], Theorem 1.1, or [11], Theorem 1.1). In finite-dimensional markets, also the conditions (NFL) and

(NFLVR) are equivalent, as shown by Delbaen and Schachermayer [6]. Thus, the absence of elementary arbitrage strategies implies that $\mathcal{M}_e^n$ is nonempty for all $n$. A sufficient condition is of course

**Assumption 2.1.** *The set $\mathcal{M}_e$ is not empty.*

and we shall stick to this assumption throughout the rest of this paper.

*Remark* 2.2. The (NFL) assumption in particular implies that the entire market, considered as a sequence of finite-dimensional markets, is free of asymptotic arbitrage and strong asymptotic arbitrage opportunities of both the first and second kind. See Kabanov and Kramkov [12, 13] and Klein and Schachermayer [17] for details. We also recall that the condition of No Asymptotic Arbitrage of the first kind is equivalent to the (NFLVR) condition. However, the equivalence between (NFL) and (NFLVR) may not necessarily hold in the case of infinitely many assets, and we refer to Klein [16] (Example 5.2) for a counterexample in the context of a general large financial market. This counterexample involves a sequence of different probability spaces, and therefore does not apply to our setting, but it indicates that the (NFL) condition is a safer choice in presence of infinitely many assets.

As already pointed out in [2], [4], the class $\mathcal{H}$ of strategies is not satisfactory, since in a large market we should admit the (theoretical) possibility of investing in infinitely many assets. In particular, an economic agent may invest istantaneously in a finite number of assets, but the global strategy may involve all the assets in the market. This idea leads to the notion of "generalized" strategy (see for instance [4]), as the limit of strategies which are pointwise finite-dimensional.

Also, the space $C_0$ in general fails to be closed in any reasonable sense, ad therefore is not suitable for optimization problems. So it needs to be enlarged in some sense, by adding some proper "limit" claims.

To this aim, it is natural to introduce a definition of stochastic integral with respect to a sequence of semimartingales.

Such a definition has been introduced in [5], to which we refer for all details and main results. Here, we recall the main definitions. For simplicity, we denote by $E = \mathbb{R}^{\mathbb{N}}$ the set of all real sequences and by $E'$ its topological dual, which is the set of linear combination of Dirac measures on $\mathbb{N}$. We call simple integrand a $E'$-valued process of the form $H = \sum_{i\leq n} h^i\delta_i$, where, as usual, $\delta_i$ denotes the Dirac delta at point $i$, and $h^i$ are bounded and predictable processes. For a simple integrand, it is naturally defined the stochastic integral with respect to $\mathbf{S}$, as

$$\int H d\mathbf{S} = \int \sum_{i\leq n} h^i dS^i$$

which trivially reduces to a finite-dimensional stochastic integral. This corresponds, in financial terms, to the definition of elementary strategies and portfolios, as recalled in Definition 2.1 ii)

A generalized integrand will be obtained as the limit, in some sense to be specified, of simple integrands, as well as a generalized strategy is the limit of finite-dimensional strategies. With the observation that a simple integrand takes values in the set of continuous operators on $E$, the results by Métivier and Pistone on stochastic integration for Hilbert-valued martingales [25] suggest that the space of integrands may contain processes with values in the set of not-necessarily bounded operators on $E$, which satisfy a proper measurability condition. We consider the semimartingale topology introduced by Emery [8] on the space of all real semimartingales (see also [22]).

**Definition 2.3.**

i) A process $\mathbf{H}$ with values in the set of non necessarily bounded operators on $E$ is *predictable* if there exists a sequence $(H^n)$ of simple processes, such that

$$\mathbf{H} = \lim_{n\to\infty} H^n,$$

in the sense that for all $x$ in the domain of $\mathbf{H}$, the sequence $H^n(x)$ converges to $\mathbf{H}(x)$, as $n$ tends to $\infty$.

ii) A predictable process $\mathbf{H}$ with values in the set of non necessarily bounded operators on $E$ is *integrable* with respect to $\mathbf{S}$ if there exists a sequence $(H^n)$ of simple integrands such that $H^n$ converges to $\mathbf{H}$ pointwise and the sequence of semimartingales $(H^n \cdot \mathbf{X})$ converges to a semimartingale $Y$ in the semimartingale topology.
In this case, we define $\int \mathbf{H} d\mathbf{S} = \mathbf{H} \cdot \mathbf{S} = Y$.

Of course, Definition 2.3 ii) makes sense if we can prove that the limit defined above is uniquely determined, namely that it is independent of the approximating sequence: the proof of this fact is provided in [5] (Proposition 5.1).

**Definition 2.4.** A *generalized strategy* is a process $\mathbf{H}$ which is integrable with respect to the semimartingale $\mathbf{S}$.

As in the finite-dimensional market, a self-financing portfolio is described by a pair $(x, \mathbf{H})$ where the constant $x$ is the initial value of the portfolio and $\mathbf{H}$ a generalized strategy. The value process of such a portfolio is defined by the formula

$$V_t = x + \int_0^t \mathbf{H}_s d\mathbf{S}_s$$

(see [4], for a more detailed discussion). As we seek to extend the definition of admissibility from elementary to general strategies, we have at least four seemingly reasonable possibilities:

**Definition 2.5.** Given $x > 0$, consider the following spaces of generalized strategies:

i) $\mathbf{H} \in \mathcal{A}_x^1$ iff there exists a sequence $\{H^n\}_{n=1}^\infty$, such that $H^n \in \mathcal{H}^n$ and is $x$-admissible, and $(H^n \cdot \mathbf{S}) \to (\mathbf{H} \cdot \mathbf{S})$ in the semimartingale topology.
ii) $\mathbf{H} \in \mathcal{A}_x^2$ iff there exists a sequence $\{H^n\}_{n=1}^\infty$ and a constant $c > 0$, such that $H^n \in \mathcal{H}^n$ and is $(x + c)$-admissible, $x + (\mathbf{H} \cdot \mathbf{S})_t \geq 0$ a.s., and $(H^n \cdot \mathbf{S}) \to (\mathbf{H} \cdot \mathbf{S})$ in the semimartingale topology.
iii) $\mathbf{H} \in \mathcal{A}_x^3$ iff $x + (\mathbf{H} \cdot \mathbf{S})$ is a nonnegative $Q$-supermartingale for all $Q \in \mathcal{M}_e$.
iv) $\mathbf{H} \in \mathcal{A}_x^4$ iff $x + (\mathbf{H} \cdot \mathbf{S})_t \geq 0$ a.s. for all $t$.

Let us comment on the economic interpretation of these definitions. The set $A_x^1$ contains those strategies which can be approximated by elementary strategies, each of them admissible with capital $x$. The set $A_x^2$ somewhat relaxes this requirement, allowing approximating strategies to remain admissible with a possibly higher capital $x + c$, while keeping the limit strategy $x$-admissible. The definition of the set $A_x^3$ is given in the spirit of optional decomposition theorems (see [9, 18]), and is the most convenient for convex duality methods, but it lacks a clear economic meaning. Finally, $A_x^4$ contains all generalized strategies which never drop below zero, without any condition on their elementary approximations.

We have the following inclusions:

$$\mathcal{A}_x^1 \subset \mathcal{A}_x^2 \subset \mathcal{A}_x^3 \subset \mathcal{A}_x^4$$

The first and the last ones are trivial, while the middle inclusion is established as in the finite-dimensional case:

**Lemma 2.1.** *Let $\mathbf{H} \in \mathcal{A}_x^2$ be an admissible strategy. Then $\mathbf{H} \cdot \mathbf{S}$ is a supermartingale for all $Q$ which make all assets $\{S^i\}_{i=1}^\infty$ local martingales.*

*Proof.* Let $H^n$ be an approximating sequence, so that $H^n \cdot \mathbf{S} \geq -(x + c)$ and $H^n \cdot \mathbf{S}$ converges to $\mathbf{H} \cdot \mathbf{S}$ in the semimartingale topology. This topology implies convergence in probability, and up to a subsequence, almost surely. The supermartingale property of $\mathbf{H} \cdot \mathbf{S}$ then follows by the the usual application of Fatou's Lemma. □

With a finite number of assets, all above classes coincide with the usual set of admissible strategies. In general, this is not the case, as the set $\mathcal{A}_x^4$ may strictly contain $\mathcal{A}_x^3$. As we are looking for a reasonable class of admissible strategies, we need to understand the properties of different classes of strategies, and compare them to our expectations.

We expect that a good definition of admissibility leads to the following properties:

i) Assumption 2.1 excludes arbitrage opportunities.
ii) Claims superreplicable with a fixed capital admit a dual characterization.

iii) The maximum expected utility on the entire market is the limit of maximum expected utility on finite-dimensional submarkets.

In some sense, this paper aims at finding such a definition, and checking that it satisfies the desired properties.

It is immediately seen that under assumption 2.1 the classes $\mathcal{A}_x^1, \mathcal{A}_x^2$ and $\mathcal{A}_x^3$ are free of arbitrage opportunities, while the class $\mathcal{A}_x^4$ poses a more delicate question. One would like to argue that, if $\mathbf{S}$ is a infinitely-dimensional local martingale under $Q \in \mathcal{M}_e$, $\mathbf{H}$ is a $\mathbf{S}$-integrable process, and $\mathbf{H} \cdot \mathbf{S}$ is bounded from below, then $\mathbf{H} \cdot \mathbf{S}$ is a local martingale, and hence a supermartingale: this is a well-known result in the finite-dimensional setting, due to Ansel and Stricker [1].

However, it turns out that with infinite dimensions the local martingale property may be lost even for stochastic integrals bounded from below.

Example 6.2 in [5] in a continuous-time setting and Example 5.1 below in the elementary case of one-period models, show that arbitrage opportunities may indeed appear in the class $\mathcal{A}_x^4$, even if Assumption 2.1 holds.

*Remark* 2.3. As already observed in [5], if every $S^i$ is a continuous semimartingale, the result by Ansel and Stricker holds even in the infinite-dimensional setting: it is a consequence of the closedness of the set of continuous local martingales in the semimartingale topology ([22], Theorem IV.5).

In this case, $\mathcal{A}_x^4 = \mathcal{A}_x^3 = \mathcal{A}_x^2$. Indeed, let $\mathbf{H} \in \mathcal{A}_x^4$ and let $H^n$ be a sequence of simple integrands such that $H^n \cdot \mathbf{S}$ converges to $\mathbf{H} \cdot \mathbf{S}$. For fixed $c > 0$, the sequence of stopping times

$$T_n = \inf \left\{ t : \int_0^t H_s^n d\mathbf{S}_s < -x - c \right\}.$$

is such that $\lim_n \mathbb{P}(T_n < T) = 0$. So, possibly up to a subsequence, we can assume that $\sum_n \mathbb{P}(T_n < T) < \infty$. The sequence of stopping times $S_n = \inf_{m \geq n} T_m$ converges to $T$ a.s. Hence, defining $\tilde{H}^n = \mathbf{1}_{[0,S_n]} H^n$, we find a sequence of simple integrands such that $\tilde{H}^n \cdot \mathbf{S} \geq -x - c$ and $\tilde{H}^n \cdot \mathbf{S}$ converges to $\mathbf{H} \cdot \mathbf{S}$, which proves that $\mathbf{H} \in \mathcal{A}_x^2$.

In fact, it can be proved that under Assumption 2.1, $A_x^1 = A_x^4$ as well, but the proof is rather technical. We omit it, since Proposition 4.1 below makes it unnecessary to our aims.

## 3. Dual characterization of superreplicable claims

By the basic superreplication result (see, for instance [6], [7]) in finite-dimensional markets, we have that, for any $X \in L_+^0$ and $x > 0$:

$$\sup_{Q \in \mathcal{M}_e^n} E_Q [X] \leq x \iff X \leq x + (H \cdot \mathbf{S})_T \text{ for some } H \in \mathcal{H}^n. \tag{3.1}$$

and we denote by $\pi_n(X)$ the superreplication price of $X$ using the first $n$ securities:

$$\pi_n(X) = \sup_{Q \in \mathcal{M}_e^n} E_Q [X] .$$

As we consider the entire market, we have two possible analogues for the left-hand side in (3.1):

$$\pi_\infty(X) = \lim_n \pi_n(X) = \inf_{n \geq 1} \sup_{Q \in \mathcal{M}_e^n} E_Q [X]$$

$$\pi(X) = \sup_{Q \in \mathcal{M}_e} E_Q [X] .$$

It is clear from the definition that the following inequality holds:

$$\pi(x) \leq \pi_\infty(x)$$

Examples 5.3 and 5.4 below show that, in some cases, $\pi(x) < \pi_\infty(x)$.

A simple observation is that, since $\pi_n$ is the superreplication price using the first $n$ securities, then $\pi_\infty(X)$ is the superreplication price obtained using only elementary strategies, as we prove in the following:

**Lemma 3.1.** *Let $X \in L^0_+$. We have that:*

$$\pi_\infty(X) = \inf\{x \,|\, X \leq x + (H \cdot \mathbf{S})_T, \text{ for some } H \in \mathcal{H}\}$$

*Proof.* The right-hand side is clearly smaller than $\pi_\infty(X)$. The reverse inequality is obtained as follows: for any $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that $\pi_n(X) < \pi_\infty(X) + \varepsilon$, and hence some $H \in \mathcal{H}^n$ such that $X \leq \pi_n(X) + (H \cdot \mathbf{S})_T$. The claim trivially follows. □

We point out that, in general, $\pi_\infty(X)$ is an infimum and not a minimum. It is now natural to ask if $\pi(X)$ can be characterized in a similar way, using a proper class of admissible generalized strategies in the right-hand side in (3.1). In fact, the following result holds:

**Theorem 3.2.** *Let $X \in L^0_+$ and $x > 0$. The following conditions are equivalent:*

*i)* $\sup_{Q \in \mathcal{M}_e} E_Q[X] \leq x$;

*ii) There exists* $\mathbf{H} \in \mathcal{A}^1_x$, *such that*

$$X \leq x + (\mathbf{H} \cdot \mathbf{S})_T.$$

In order to prove Theorem 3.2, we need first to introduce some notation and preliminary results. For a set $A \subset L^0$, we denote by $\overline{A}^p$ its closure in the space $L^0$, endowed with convergence in probability. Following Kramkov and Schachermayer [19], we introduce the sets

$$\mathcal{C}_n = \{X \in L^0_+ : X \leq 1 + (H \cdot \mathbf{S})_T,\, H \in \mathcal{H}^n\} = 1 + C_{0,n},$$

and $\mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}_n$. Recall that the polar of a set $A \subset L^0_+$ is defined by:

$$A^\circ = \{f \in L^0_+ : E\,[fg] \leq 1 \text{ for all } g \in A\}.$$

Next lemma is quite standard: we include its short proof for completeness, since it does not follow directly from analogous results with finitely many assets.

**Lemma 3.3.** *Let $X \in L^0_+$. Then*

$$\sup_{Q \in \mathcal{M}_e} E_Q[X] = \sup_{f \in \mathcal{C}^\circ} E[fX]$$

*Proof.* Note first that $\mathcal{M}_e \subset \mathcal{C}^\circ$: in fact, for any admissible $H \in \mathcal{H}$, we have that $(H \cdot \mathbf{S})$ is a supermartingale, and hence $E_Q[1 + (H \cdot \mathbf{S})_T] \leq 1$. This proves one inequality.

Viceversa, suppose that $f \in \mathcal{C}^\circ$. Since $1 \in \mathcal{C}$ and $-L^\infty_+ \subset \mathcal{C}$, it follows that $f \in L^1_+$ and that $E[f] \leq 1$. Let $\{f_n\}_{n=1}^\infty$ be a maximizing sequence for the right-hand side. Up to a rescaling, which only increases the expectation, we can assume that $E[f_n] = 1$ for all $n$. By assumption 2.1, we can consider $Q \in \mathcal{M}_e$, and denote by $g = \frac{dQ}{dP}$ its density. We define a new sequence of measures $\{Q_n\}_{n=1}^\infty$, defined by $\frac{dQ_n}{dP} = (1 - \frac{1}{n})f_n + \frac{1}{n}g$. It is clear that $Q_n \in \mathcal{M}_e$ for all $n$, and that $\lim_{n\to\infty} E_{Q_n}[X] = \lim_{n\to\infty} E[f_n X]$. □

The following result holds under the weaker assumption that condition (NFLVR) is satisfied by the market:

**Lemma 3.4.** *Let $(f_n)$ be a sequence of random variables, such that $-1 \leq f_n \leq (H^n \cdot \mathbf{S})_T$, where $H^n$ are admissible elementary strategies. Assume that $f_n$ converges almost surely to $f$. Then, there exists a process $\mathbf{H} \in \mathcal{A}^1_1$ such that $f \leq (\mathbf{H} \cdot \mathbf{S})_T$.*

*Proof.* The proof is essentially contained in Section 4 in [6] (see also [11], section 2 and 3, for an alternative proof). We only give a sketch of the main steps. Let us start by observing that $\mathcal{M}^n_e \neq \emptyset$, since (NFLVR) holds also in the market based on the first $n$ securities. It follows easily that $(H^n \cdot \mathbf{S})_t \geq -1$ for all $t \leq T$, namely, $H^n$ is 1-admissible. Let us denote by $K^1_0 = \{(H \cdot \mathbf{S})_T : H \in \mathcal{H}, H \text{ 1-admissible}\}$. By Lemma A1.1 in [6], there exists a sequence of convex combinations $(\tilde{H}^n) \in \mathrm{conv}(H^n, H^{n+1}, \ldots)$, such that $(\tilde{H}^n \cdot \mathbf{S})_T$ converges almost surely. Notice that $\tilde{H}^n$ are still 1-admissible elementary strategies. It follows that $f \leq g$, where $g$ is some element in $\overline{K^1_0}^p$, hence the set $\mathcal{D}_f = \{g \in \overline{K^1_0}^p : g \geq f \text{ a.s.}\}$ is not empty. Since $K^1_0$ is bounded in $L^0$ because of the (NFLVR) assumption (see [6], Corollary 3.4 or [11], Lemma 2.2), it follows that $\mathcal{D}$ is also bounded. Lemma 4.3 in [6] implies that $\mathcal{D}_f$ contains a maximal element, denoted by $f_0$, which can be written in the form $f_0 = \lim_n (L^n \cdot \mathbf{S})_T$, where $L^n$ are 1-admissible elementary strategies and the convergence is in probability. It can be easily checked that the set of elementary strategies which we are considering satisfies the hypotheses of stability with respect to the operations carried out by Delbaen and Schachermayer in section 4 of [6], or by Kabanov in [11]. So, their results can be applied: we refer in particular to Lemmas 4.5, 4.10 and 4.11 in [6] (also the results in section 2 and 3 in [11]). It follows that there exist

$(\tilde{L}^n) \in \mathrm{conv}(L^n, L^{n+1}, \ldots)$ such that the sequence of semimartingales $(\tilde{L}^n \cdot \mathbf{S})$ is a Cauchy sequence in the semimartingale topology.

At this point, the main difference between the case considered in [6], [11] and ours, comes out. In [6], [11], since a finite-dimensional market is considered, all strategies are 1-admissible with respect to a $d$-dimensional semimartingale $S = (S^i)_{i \leq d}$, where $d$ is fixed, and $(\tilde{L}^n \cdot S)$ is a standard stochastic integral in $\mathbb{R}^d$. In this case, it can be applied a result by Mémin ([22], Corollay III.4), which claims that there exists a predictable $S$-integrable process $L$, such that $(\tilde{L}^n \cdot S)$ converges to $L \cdot S$.

In our framework, we can still represent the limit of $(\tilde{L}^n \cdot \mathbf{S})$ as a stochastic integral, but, in this case, we need a generalized strategy: by Theorem 5.2 in [5], there exists a generalized strategy $\mathbf{H}$ such that $(\tilde{L}^n \cdot \mathbf{S})$ converges to $\mathbf{H} \cdot \mathbf{S}$. It is evident that $\mathbf{H} \in \mathcal{A}_1^1$. □

The previous lemma is useful in order to characterize the closure of $\mathcal{C}$.

**Lemma 3.5.** *The following result holds:*

$$\overline{\mathcal{C}}^p = \{X \in L^0_+ : X \leq 1 + (\mathbf{H} \cdot \mathbf{S})_T,\ \mathbf{H} \in \mathcal{A}_1^1\}$$

*Proof.* Let $(X^n)_n$ be a sequence in $\mathcal{C}$, converging in probability to a random variable $X$. Up to a subsequence, we can assume that $X^n$ converges almost surely to $X$. Then, Lemma 3.4 applied to the sequence $(X^n - 1)$ shows that there exists a generalized strategy $\mathbf{H} \in \mathcal{A}_1^1$ such that $X \leq 1 + (\mathbf{H} \cdot \mathbf{S})_T$.

Conversely, assume that $X \leq Y = 1 + (\mathbf{H} \cdot \mathbf{S})_T$ for some $\mathbf{H} \in \mathcal{A}_1^1$. The random variable $Y$ belongs to $\overline{\mathcal{C}}^p$, which is solid (we recall that a subset $A \subset L^0_+$ is called solid if $g \in A, h \in L^0$ and $0 \leq h \leq g$ implies that $h \in A$). It follows that $X \in \overline{\mathcal{C}}^p$. □

We are now ready to prove Theorem 3.2

*Proof of Theorem 3.2.* We can assume, without loss of generality, that $x = 1$. By Lemma 3.3, condition $i)$ amounts to say that $X$ belongs to $\mathcal{C}^{\circ\circ}$, which is the bipolar of $\mathcal{C}$. By the bipolar theorem (in the version due to Brannath and Schachermayer [3]), $\mathcal{C}^{\circ\circ}$ is the closed convex solid hull of $\mathcal{C}$ in $L^0_+$. Since $\mathcal{C}$ is convex and solid, $\mathcal{C}^{\circ\circ}$ is just the closure of $\mathcal{C}$ in $L^0$. Then, the equivalence between $i)$ and $ii)$ follows from the characterization of $\overline{\mathcal{C}}^p$ given in Lemma 3.5 □

*Remark* 3.1. It is not difficult to check, using the definition of the set $\mathcal{A}_x^1$ and Lemma 3.4, that condition $ii)$ is equivalent to the following

$ii)'$ There exists a sequence $\{H^n\}_{n=1}^{\infty}$, such that $H^n \in \mathcal{H}^n$ and is $x$-admissible, and

$$X \leq x + P\text{–}\lim_{n\to\infty} (H^n \cdot S)_T.$$

We recall indeed that convergence in the semimartingale topology implies convergence in probability of the terminal values. Condition $ii)'$ makes more sense from a financial point of view, since it involves "real" (that is, finite-dimensional) strategies. Hovewer, the statement of Theorem 3.2 gives us a relation analogous to (1.2), which allows us to immediately apply the already existing results on utility maximization (see next section).

## 4. Utility Maximization

We consider the problem of utility maximization: as in [19], let $U$ be a utility function such that $U(x) = -\infty$ for $x < \infty$ and saitysfying the so-called Inada conditions. As usual, $V$ will denote the conjugate function of $U$, namely

$$V(y) = \sup_{x>0}[U(x) - xy]$$

for $y > 0$. It is well-known that $V$ satisfies the inversion formula

$$U(x) = \inf_{y>0}[V(y) + xy]$$

(see [19] for further details). For all $n \geq 1$, we define the problem

$$u_n(x) = \sup_{H \in \mathcal{H}^n} \mathbb{E}\left[U\left(x + \int_0^T H_s d\mathbf{S}_s\right)\right] = \sup_{X \in x\mathcal{C}_n} \mathbb{E}\left[U(X)\right] \tag{4.1}$$

We denote by $\mathcal{D}_n$ the polar of $\mathcal{C}_n$: this set was characterized by Kramkov and Schachermayer as the closed, convex, solid hull of the set $\mathcal{M}_e^n$ (see[19] for details). The dual problem of (4.1) is then defined by:

$$v_n(y) = \inf_{Y \in \mathcal{D}_n} \mathbb{E}\left[V(yY)\right]. \tag{4.2}$$

If we assume that $u_n(x) < \infty$ for all $x$, the function $v_n(y)$ is the convex conjugate of $u_n(x)$ ([19], Theorem 2.1).

Let us define

$$u_\infty(x) = \lim_n u_n(x) \qquad\qquad v_\infty(y) = \lim_n v_n(y);$$

clearly, $u_\infty(x) = \sup_{H \in \mathcal{H}} \mathbb{E}\left[U\left(x + \int_0^T H_s d\mathbf{S}_s\right)\right]$, that is, $u_\infty(x)$ is the value function of the utility maximization problem over all the elementary strategies. To eliminate trivial cases, we assume that $u_\infty(x_0) < \infty$ for some $x_0 > 0$. Since $u_\infty$ is is increasing and concave (as limit of an increasing sequence of increasing and concave functions), it follows that $u_\infty(x) < \infty$ for all $x > 0$.

We wish now to consider the problem of maximizing expected utility over a class of generalized strategies: we can define three different problems according to the three classes $\mathcal{A}^i_x, i = 1, 2, 3$. However, an important consequence of Theorem 3.2 is the following:

**Proposition 4.1.** *We have that:*

$$\overline{\mathcal{C}}^p = \{X \in L^0_+ : X \leq x + (\mathbf{H} \cdot \mathbf{S})_T,\, \mathbf{H} \in \mathcal{A}^3_x\}$$

*Proof.* Let $\mathbf{H} \in \mathcal{A}^3_x$. Then it clearly satisfies $i)$ in Theorem 3.2, and therefore $(\mathbf{H}\cdot\mathbf{S})_T$ is dominated by some $(\mathbf{K}\cdot\mathbf{S})_T$, with $\mathbf{K} \in \mathcal{A}^1_x$. The reverse inclusion is a trivial consequence of Lemma 3.5. □

The above proposition implies that, without loss of generality, we may define over the class $\mathcal{A}^1$ both the utility maximization problem:

$$\max_{\mathbf{H}\in\mathcal{A}^1_x} E\left[U\left(x + (\mathbf{H}\cdot\mathbf{S})_T\right)\right] \tag{4.3}$$

and its value function:

$$u(x) = \sup_{\mathbf{H}\in\mathcal{A}^1_x} E\left[U\left(x + (\mathbf{H}\cdot\mathbf{S})_T\right)\right]$$

Since $\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$, it is easy to check that $\mathcal{C}^\circ = \bigcap_{n\geq 1} \mathcal{D}_n = \mathcal{D}$: then, the dual value function is defined as follows:

$$v(y) = \inf_{f\in\mathcal{D}} E\left[V\left(yf\right)\right] \tag{4.4}$$

The inequalities $u(x) \geq u_\infty(x)$ and $v(y) \geq v_\infty(y)$ are evident: in fact, we will prove that equality holds in both cases. Let us start by proving the second one:

**Lemma 4.2.** $v(y) = v_\infty(y)$*, for all* $y > 0$*.*

*Proof.* Denote by $v_\infty(y) = \sup_n v_n(y)$. We have to show that $v(y) \leq v_\infty(y)$, as the reverse inequality is trivial. Let $Y_n \in \mathcal{D}_n$ be such that

$$\lim_{n\to\infty} E\left[V(yY_n)\right] = v_\infty(y)$$

By Lemma A1.1 in [6] (see also Lemma 3.3 in [19]), there exists a sequence $(Z_n) \in \mathrm{conv}(Y_n, Y_{n+1}, \ldots)$, which converges almost surely: note that $Z_n \in \mathcal{D}_n$ and therefore $Z = \lim_n Z_n = \in \cap_{n=1}^\infty \mathcal{D}_n = \mathcal{D}$. By the convexity of $V$, it is easy to verify that

$$\lim_{n\to\infty} E\left[V(yZ_n)\right] = v_\infty(y)$$

Finally, Lemma 3.4 in [19] implies that the sequence $(V^-(yZ_n))$ is uniformly integrable, hence

$$v(y) \leq E\left[V(yZ)\right] \leq \liminf_{n\to\infty} E\left[V(yZ_n)\right] = v_\infty(y)$$

□

**Lemma 4.3.** *There exists $y_0$ such that $v(y) < \infty$ for $y > y_0$.*

*Proof.* For all $n$, the following relation holds for $y > 0$:

$$v_n(y) = \sup_{x>0}(u_n(x) - xy)$$

(see [19], Theorem 3.1), hence

$$v(y) \leq \sup_{x>0}(u_\infty(x) - xy). \tag{4.5}$$

Since $u_\infty$ is concave, the thesis easily follows. □

Now, we wish to prove that $u(x) = u_\infty(x)$. Consider first the case when $U(x) \geq -M$ for all $x \geq 0$: each element of $\overline{\mathcal{C}}$ is limit of a sequence of elements in $\cup_n \mathcal{C}_n$; then, the claimed equality is a consequence of Fatou's lemma. In the case when $\lim_{x\to 0^+} U(x) = -\infty$, this argument does not work. It is evident that the equality holds when $u_\infty(x) = \infty$. If we assume, as above, that $u_\infty(x) < \infty$ for all $x$, then, a proof of this relation can be obtained by exploiting the duality between $u$ and $v$.

**Proposition 4.4.** *$u_\infty(x) = u(x)$, for all $x > 0$.*

*Proof.* Let $X \in x\overline{\mathcal{C}}^p, Y \in y\mathcal{D}$: the inequality $U(X) \leq V(Y) + XY$ implies, by a simple integration, that $u(x) \leq v(y) + xy$ for all $y > 0$, namely,

$$u(x) \leq \inf_{y>0}(v(y) + xy).$$

In particular, it follows that $u(x) < \infty$ for all $x > 0$. We can then apply Theorem 3.1 in [19], to prove that $u$ and $v$ are in duality and $v$ is the convex conjugate of $u$: precisely,

$$u(x) = \inf_{y>0}(v(y) + xy)$$

$$v(y) = \sup_{x>0}(u(x) - xy).$$

Denote by $\tilde{v}$ the convex conjugate of $u_\infty$. Since (4.5) holds, we have that $\tilde{v}(y) \geq v(y)$ for all $y$. So, we obtain $u_\infty(x) \geq u(x)$, which completes the proof. □

At this point, using the argument of the proof of Lemma 3.5 in [19], with hardly any modification, we can prove that for all $y$ which satisfy the condition $v(y) < \infty$, there exists a minimizing element for the problem (4.4), that is, there exists $\hat{h}(y)$ such that

$$v(y) = \mathbb{E}\left[V(y\hat{h}(y))\right].$$

However, there does not necessarily exist an optimizer for the utility maximization problem (4.3). From the results of [20], we obtain that, in order to prove the existence of a maximizer, we need the further hypothesis that

**Assumption 4.1.** *$v(y) < \infty$ for all $y > 0$.*

We point out that this condition is implied by the condition on the asymptotic behaviour of the elasticity of $U$:

$$AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1$$

(see [19, 20] for details). The most general results under the Assumption 4.1 are resumed in the next theorem, which follows from the results in [20] and from what we have already proved.

**Theorem 4.5.** *If Assumptions 2.1 and 4.1 hold true, then we have that:*

*i) The value functions $u$ and $v$ are continuously differentiable, increasing and strictly concave on $(0, \infty)$ and satisfy Inada conditions.*

*ii) The optimal solution $\hat{X}(x) = (\hat{\mathbf{H}}(x) \cdot \mathbf{S})_T$ to (4.3) exists for any $x > 0$, and is unique in the sense that the terminal wealth $\hat{X}(x)$ is unique. In addition, if $y = u'(x)$, we have that $U'(\hat{X}_T(x)) = \hat{Y}_T(y)$, where $\hat{Y}(y)$ is the optimal solution to (4.4).*

*iii) The function $v$ satisfies the representation:*

$$v(y) = \inf_{Q \in \mathcal{M}_e} E\left[V\left(y\frac{dQ}{dP}\right)\right]$$

*iv) The following relation holds:*

$$u(x) = \sup_{H \in \mathcal{H}} E\left[U(x + (H \cdot \mathbf{S})_T)\right]$$

## 5. (Counter)examples

In this section we illustrate with examples some phenomena which may arise in presence of infinitely many assets. To stress that these are basic issues, as opposed to so-called “mathematical pathologies”, in most cases we shall deal with elementary, one-step Arrow-Debreu models. It should be noticed, however, that such models have jumps, and in fact some of the couterexamples disappear in the case of continuous processes (see also Remark 2.3).

The first example shows a one-period model with infinitely many assets, with the following feature: if the agent is allowed to invest in a finite number of securities, his optimal choice is not to invest at all. On the other hand, were he allowed to invest in all available assets simultaneously, the optimal choice would be to invest in the first asset, and hedge some of the risk by investing in all other assets. This phenomenon arises for a simple reason: all assets except the first one are martingales, that is bad investments (by themselves). On the contrary, the first asset offers a positive return, but it has a large downside risk, which cannot reasonably hedged (i.e. turned into an admissible strategy) with finitely many securities.

**Example 5.1.** Consider a one-period model (i.e. $T = \{0,1\}$) on the countable probability space $\Omega = \{\omega_n\}_{n=0}^{\infty}$. $\mathcal{F}_0$ and $\mathcal{F}_1$ are respectively the trivial and the discrete $\sigma$-algebra on $\Omega$. The probability and the assets prices at the final time 1 are defined by the following table, where $\alpha \in (0,1)$ and $\beta > 0$.

| $\omega$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\dots$ | $\omega_n$ |
|---|---|---|---|---|---|---|
| $P(\omega)$ | $1-\alpha$ | $\alpha 2^{-1}$ | $\alpha 2^{-2}$ | $\alpha 2^{-3}$ | $\dots$ | $\alpha 2^{-n}$ |
| $S_1^1$ | $-1$ | $\beta$ | $\beta^2$ | $\beta^3$ | $\dots$ | $\beta^n$ |
| $S_1^2$ | 0 | 0 | $\beta^2$ | 0 | $\dots$ | 0 |
| $S_1^3$ | 0 | 0 | 0 | $\beta^3$ | $\dots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $S_1^n$ | 0 | 0 | 0 | 0 | $\dots$ | $\beta^n$ |

In other words, $P(\omega_0) = 1-\alpha$ and $P(\omega_n) = \alpha 2^{-n}$ for $n \geq 1$. For the first asset, $S_1^1(\omega_0) = -1$ and $S_1^1(\omega_k) = \beta^k$ for $n \geq 2$. For all other assets, $S_1^n(\omega_n) = \beta^n$ and $S_1^n(\omega_k) = 0$ for all $k \neq n$. Note that by choosing $\alpha$ small, we can assume that $E\left[S_1^1\right] < 0$.

Finally, we set initial prices as follows: $S_0^1 = 0$ and $S_0^n = \alpha 2^{-n}\beta^n$ for $n \geq 2$. By choosing $\beta < 2$, we assume that $S_0^n$ decreases to zero. These choices imply that $S_1^1$ is a supermartingale, while $S_0^n$ are martingales for $n \geq 2$.

Note that this market is arbitrage-free, as one can also make $S^1$ a martingale, by adjusting $P(\omega_0)$ and $P(\omega_1)$, which does not affect the martingale property of $\{S^n\}_{n\geq 2}$.

If we consider the strategies based on the first $n$ assets, the solution to the utility maximization problem is trivial. In fact, the investment on the first asset must be nonnegative, otherwise the strategy is not admissible (as $S_1^1$ is unbounded from above). On the other hand, $S^1$ is a supermartingale, therefore the optimal choice is a null position. Since all other assets are martingales, it is also optimal not to invest in any of them. It follows that the maximum utility is given by $U(x)$, where $x$ is the initial capital of the agent.

However, if we allow the agent to invest on all the securities, this situation changes. Consider the strategy of short-selling 1 unit of $S^1$, and buying 1 unit of each of the securities $\{S^n\}_{n\geq 2}$. This strategy is clearly admissible, as its support is $\{-\beta, 1\}$, its cost is given by $\sum_{n=1} \alpha 2^{-n}\beta^n = \frac{\alpha\beta}{2-\beta}$, and its expected valued is $1 - \alpha - \alpha\beta(\frac{1}{2} + \frac{1}{2-\beta})$.

Again, if $\alpha$ and $\beta$ are small enough, adding to any initial capital $x$ some small multiple of this strategy will certainly increase expected utility (as long as it is smooth enough that the first-order Taylor expansion holds).

This shows that the maximum utility is strictly greater than $U(x)$, and therefore it is not the limit of the finite-dimensional maxima, which are all $U(x)$.

The next example, which is a minor variation of the previous one, shows that if too many strategies are allowed, not only utility maximization does not pass to the limit, but even arbitrage can arise.

**Example 5.2.** In the previous example, set $P(\omega_0) = 0$, and renormalize the other probabilities accordingly. All assets $S^n$, with $n > 1$, can be made martingales rescaling prices, while the first $S^1$ loses its downside risk, and cannot become a martingale at price zero.

In fact, the same strategy as in the previous example delivers $\beta$ with certainty, and costs only a multiple of $\frac{\alpha\beta}{2-\beta}$, which can be made less than $\beta$ by choosing $\alpha$ small enough. It follows that performing this strategy after borrowing its cost from the riskless asset delivers an arbitrage.

The next two examples show that, with infinitely many assets, the superreplication prices $\pi(X)$ and $\pi_\infty(X)$ may be different.

**Example 5.3.** Consider a one-period model (i.e. $T = \{0, 1\}$) on the countable probability space $\Omega = \{\omega_n\}_{n=0}^\infty$. $\mathcal{F}_0$ and $\mathcal{F}_1$ are respectively the trivial and the discrete $\sigma$-algebra on $\Omega$. The probability and the assets prices at the final time 1 are defined by the following table, where $\alpha \in (0, 1)$ and $\beta > 0$.

| $\omega$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\ldots$ | $\omega_n$ |
|---|---|---|---|---|---|---|
| $P(\omega)$ | $1-\alpha$ | $\alpha 2^{-1}$ | $\alpha 2^{-2}$ | $\alpha 2^{-3}$ | $\ldots$ | $\alpha 2^{-n}$ |
| $S_1^1$ | 1 | 1 | 0 | 0 | $\ldots$ | 0 |
| $S_1^2$ | 1 | 0 | 2 | 0 | $\ldots$ | 0 |
| $S_1^3$ | 1 | 0 | 0 | 4 | $\ldots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $S_1^n$ | 1 | 0 | 0 | 0 | $\ldots$ | $2^n$ |

In practice, $S_1^n(\omega_0) = 1$, $S_1^n(\omega_n) = 2^n$ and $S_1^n(\omega_k) = 0$ for all $k \notin \{0, n\}$. We set the inital price of all assets to some constant $c > 0$.

Note that this market is complete: to see this, it is sufficient to show that all Arrow-Debreu securities $X_j : \omega_i \mapsto \delta_{ij}$ are replicable. For $k \geq 2$, we have trivially $X^k = 2^{-k} S^k$, hence it is sufficient to replicate $X^0$ ($X^1$ will be obtained from the riskless asset by difference). Consider the strategy of borrowing one unit of the riskless asset, and holding $\{\theta_n\}_{n=1}^\infty$ units of risky assets. If we set $\theta_1 = 1$ and $\theta_n = 2^{-n}$, the payoff of this strategy will be exactly $X^0$.

Let us now consider the cost of superreplicating the claim $X^0$. If we have only a finite number of assets at our disposal, it is intuitively clear that this cost will be at least $c$. This can be seen as follows: let $Q$ be a martingale measure for all $S^n$, and denote by $q_n = Q(\omega_n)$. In the market with the first

$n$ securities, we have the system of $n$ equations in $n+1$ unknowns:

$$\begin{aligned} q_0 + q_1 &= c \\ q_0 + 2q_2 &= c \\ &\vdots \\ q_0 + 2^n q_n &= c \end{aligned}$$

which has 1-dimensional set of solutions:

$$\begin{aligned} &q_0 \in (0, c) \\ &q_k = 2^{-k}(c - q_0) \qquad 1 \leq k \leq n \end{aligned}$$

hence the supremum of $q_0$ (the price of $X^0$ under $Q$) is clearly $c$.

Note that for finite $n$ the condition $\sum_{k=1}^{\infty} q_k = 1$ remains vacuous, but this is no longer true when $n = \infty$. In this case, the only martingale measure $Q$ is given by:

$$\begin{aligned} &q_0 = 2c - 1 \\ &q_k = 2^{-k}(c - q_0) \qquad k \geq 1 \end{aligned}$$

and $2c - 1 < c$ whenever $c < 1$.

**Example 5.4.** Assume for simplicity that $T = 1$ and that the price processes evolve according to the following dynamics:

$$dS_t^i = S_{t-}^i \left( \alpha_i dt + d\hat{N}_t + dW_t^i \right)$$

where $(W^i)_{i \geq 1}$ is a sequence of independent Wiener processes and $\hat{N}_t = N_t - t$ is a compensated Poisson process with intensity 1 ($N$ is the Poisson process), independent of $W^i$ for all $i$.
We assume that $(\mathcal{F}_t)_{t \leq 1}$ is the filtration generated by the price processes, hence by $\{(W^i)_{i \geq 1}, N\}$. It is well-known that in this case, every local martingale $L$ has necessarily the form

$$L_t = L_0 + \int_0^t k_s d\hat{N}_s + \sum_{i \geq 1} \int_0^t h_s^i dW_s^i, \tag{5.1}$$

where $k, (h^i)_{i \geq 1}$ are predictable processes and

$$\int_0^1 |k_s| ds + \sum_{i \geq 1} \int_0^1 (h_s^i)^2 ds < \infty \qquad \text{a.s.} \tag{5.2}$$

Let $Q$ be a probability measure equivalent to $P$. Then, its density has the form $dQ/dP = \mathcal{E}(L_1)$ (we recall that $\mathcal{E}$ denotes the stochastic exponential), where $L$ has the form (5.1), with $L_0 = 0$; furthermore, $k_s > -1$ to ensure that $\mathcal{E}(L_1) > 0$ and $L$ is such that $\mathcal{E}(L_t)$ is a uniformly integrable martingale.

By Girsanov's theorem, it follows that the process $W^i_t - \int_0^t h^i_s ds$ is a $Q$-Wiener process, while the process $\hat{N}_t - \int_0^t k_s ds = N_t - \int_0^t (1+k_s)ds$ is a $Q$-martingale (namely $\int_0^t (1+k_s)ds$ is the $Q$-compensator of the point process $N$).

Since $(S^i)_{i\leq n}$ is locally bounded, we have that $Q \in \mathcal{M}^n_e$ if and only if $(S^i)_{i\leq n}$ is a $Q$-local martingale. This occurs if and only if

$$h^i_t = \alpha_i + k_t$$

for all $i \leq n$. A necessary condition for Assumption 2.1 to hold is that the above equality is satisfied for all $i \geq 1$. Then, by condition (5.2), it must be $\sum_i \alpha_i^2 < \infty$, $h^i_t = \alpha_i$ for all $i$ and, necessarily $k_t \equiv 0$; namely, there exists a unique equivalent martingale measure $Q$ (we notice that the uniform integrability of the density $\mathcal{E}(-\int \sum_j \alpha_j dW^j_t)$ is a consequence of Novikov condition). Conversely, on the $n$-dimensional market, there are infinitely many equivalent martingale measures. In particular, the point process $N$ may have any intensity, and, possibly, even a stochastic compensator.

Consider the claim $X = 1\!\!\mathrm{I}_{\{N_1=0\}}$. In the large market, $N_1$ is a Poisson random variable with intensity 1, hence $w = E_Q[X] = e^{-1}$. In the $n$-market, $N_1$ may be a Poisson random variable with any intensity (or, possibly, a random variable with more general distribution): it is evident, then, that $w_\infty = 1 > e^{-1}$.

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