Asymmetric Information in Fads Models

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August 31, 2004

Abstract

Fads models were introduced by Shiller (1984) and Summers (1986) as plausible alternatives to the efficient markets/constant expected returns assumptions. Under these models, logarithms of asset prices embody both a martingale component, with permanent shocks, and a stationary component, with temporary shocks.

We study a continuous-time version of these models from both the point of view of informed agents, who can observe both fundamental and market values, and from that of uninformed agents, who can only observe market prices. We specify the asset price in the larger filtration of the informed agent, and then derive its decomposition in the smaller filtration of the uninformed agent using the Hitsuda representation of Gaussian processes. For uninformed agents we obtain a non-Markovian dynamics, which justifies the use of *technical analysis* in optimal trading strategies. For both types of agents, we solve the problem of maximization of expected logarithmic utility from terminal wealth, and obtain an explicit formula for the additional logarithmic utility of informed agents.

Finally, we apply the decomposition result to the problem of testing the presence of fads from market data. An application to the NYSE-AMEX indices from the CRSP database shows that, if the fads component prevails, then the mean-reversion speed must be slow. [1]

Key words: Brownian Motion – Ornstein-Uhlenbeck – Hitsuda representation – fads – mean reversion

JEL Classification: C15,C22

Mathematics Subject Classification (2000): 60G15, 62M07, 45D05, 91B84

[1]The author wishes to thank Patrick Cheridito for suggesting a simplification in the proof of Theorem 2.1.

1 Introduction

In an efficient market, where "prices always fully reflect available information" (Fama, [Fam70]) asset prices should be martingales, after adjusting for risk. This paradigm, when combined with the assumption of constant expected returns, leads to the classical random walk models and to Geometric Brownian Motion in continuous time.

The validity of these assumptions has been questioned on a number of grounds. Le Roy and Porter [LP81] and Shiller [Shi81] have argued that the volatility observed in stock and bond markets is too high to be explained by the flow of information on fundamentals, such as dividends. To explain this excess of volatility, Shiller [Shi81] emphasized the role of investor overreactions, fashions and fads in price shocks. In this spirit, Summers [Sum86] proposed the following discrete time model for the log price p_t of a risky asset in an "inefficient" market:

$$p_t = p_t^* + u_t \tag{1}$$

$$p_t^* = p_{t-1}^* + \varepsilon_t \tag{2}$$

$$u_t = \rho u_{t-1} + \delta_t \tag{3}$$

where ε_t and δ_t are mutually independent, identically distributed innovations, with finite variances. In this model, the martingale p_t^* represents the *fundamental value* of the asset, which has permanent shocks. The difference $u_t = p_t - p_t^*$ represents the current mispricing of the assets, or 'fad', which has temporary shocks. This model provides a plausible alternative to the assumption of constant expected returns, and accounts for the long term mean-reversion observed in stock prices by Poterba and Summers [PS88].

To assess the relevance of these models, one has to address two basic questions: do fads exists? And even if they exist, do they matter? The former question is essentially a statistical problem, which requires to test the hypothesis that the variance of δ_t is null. The latter question is a theoretical one, which can be made precise as follows. For a rational agent (i.e. a utility maximizer) trading in the asset, to what extent does the presence of fads affect maximum expected utility and optimal trading strategies?

Both these problems are complicated by a common feature. In the empirical problem, the econometrician can usually observe historical market prices only, therefore cannot rely on the innovations ε_t and δ_t for statistical purposes. Similarly, the utility maximization problem may lead to very different results, depending on whether the agent can observe both fundamental and market values, or market prices only.

This paper addresses these two issues, in a continuous-time version of the fads model above. We consider two types of agents: *informed agents* who can observe both fundamental and market values, and *uninformed agents*, who can only observe market values. For both agents we study the problem of logarithmic utility maximization from terminal wealth, and obtain a closed-form solution for expected logarithmic utilities. Then we exploit the decomposition of the asset price dynamics for uninformed agents to statistically test the presence of fads from market data.

It is worthwhile to make a comparison between this model and those of insider trading recently considered, among others, by Karatzas and Pikovsky [PK96], Amendinger, Imkeller and Schweizer [AIS98], Baudoin [Bau02], Corcuera, Imkeller, Kohatsu-Higa, Nualart [CIKHN04], and Kohatso-Higa [KH04] for a recent survey. In these models, the asset price dynamics is specified under the filtration of the ordinary (uninformed) agent. Then one specifies the extra information available to the insider, usually some functional of future prices (possibly disturbed by a noise), which is eventually revealed at the final time T. The dynamics for the insider is then obtained by an enlargement of filtration (or, in [Bau02], by conditioning the stochastic differential equation).

By contrast, we specify the asset price process in the larger filtration of informed agents, and then obtain its dynamics for uninformed agents using the Hitsuda representation of Gaussian processes. We observe that the additional information of the informed agent continues to grow over time, and is never revealed. However, mean-reversion causes past additional information to gradually lose value, leading to a finite expected utility.

As we restrict the larger filtration instead of enlarging the smaller one, we note that from the viewpoint of the *uninformed agent*, the asset price loses the Markov property, and optimal investment decisions involve current as well as past prices. This allows for a suggestive interpretation on the relationship between *fundamental* and *technical* analysis: the informed agent, who has access to both fundamental and market values, does not care about the price history, and invests optimally by looking at only the current mispricing. On the other hand, the uninformed agent is aware of the presence of a mispricing, but cannot observe it directly, and uses technical analysis to extract from past prices some information about it.

The rest of the paper is organized as follows: in section 2 we describe the model in detail, and state the decomposition result, whose proof is postponed to section 5. In section 3 we solve the logarithmic utility maximization problem, derive explicit formulas for expected maximum utilities, and obtain a surprisingly simple asymptotic formula for the additional logarithmic utility, which allows to perform intuitive comparative statics.

We conclude in section 4 with the econometric application. Results for the NYSE-AMEX indices from the CRSP data base show that if the fads component of volatility prevails, as the literature on variance bounds suggests (see Le Roy and Porter [LP81] and Shiller [Shi81]), then the mean-reversion speed must be slow. This confirms and improves similar findings of Poterba and Summers [PS88] and Shiller and Perron [SP85], obtained by Monte Carlo simulations.

The Appendix resumes some technical results used in the paper.

2 The Model

We consider a model of a financial market with one riskless asset B (the *money market*) and one risky asset S (the *stock market*), and assume that the riskless asset is identically equal to one.

To describe the dynamics of the risky asset, we consider a probability space $(\Omega, \mathcal{F}, P)$, on which are defined two independent Brownian Motions $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$. We denote by U_t the Ornstein-Uhlenbeck process obtained as the unique solution to the Langevin equation:

$$U_t = -\lambda \int_0^t U_s ds + B_t \tag{4}$$

where $\lambda > 0$. We denote by

$$Y_t = pW_t + qU_t \tag{5}$$

where $p, q > 0$ and $p^2 + q^2 = 1$. We introduce two deterministic, Lebesgue measurable functions μ_t and $\sigma_t > 0$, and define the price of the risky asset S_t as the solution to the stochastic differential equation:

$$dS_t = S_t \left(\mu_t dt + \sigma_t dY_t\right) \tag{6}$$

which is given by:

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dY_s\right) \tag{7}$$

The economic interpretation of (6) is that price shocks embody both a permanent component, represented by the martingale W, and a temporary component, represented by the mean-reverting process U_t. In particular, when q or λ are null (and μ_t, σ_t are constant), we recover the usual Geometric Brownian Motion.

We introduce the two filtrations $(\mathcal{F}_t^0)_{t\in[0,\infty)}$ and $(\mathcal{F}_t^1)_{t\in[0,\infty)}$ which are the augmentations of the filtrations generated by Y (or equivalently S) and (W, B) (or equivalently (S, U)) respectively, and thus satisfy the usual conditions of right-continuity and completeness. We obviously have that $\mathcal{F}_t^0 \subset \mathcal{F}_t^1$ for all t. Our market is populated by two types of agents: *informed agents*, with access to the larger filtration $\mathcal{F}_t^1$ and *uninformed agents*, with access to $\mathcal{F}_t^0$ only. The underlying idea is that informed agents, who observe both S and U, are aware at all times of the extent to which the stock market is over (or under) valued, while uninformed agents only see the market price S.

To avoid common misconceptions, we remark that while this model is a prototype for an "inefficient" market, it is still free of arbitrage opportunities in the usual sense. In fact, it is easy to see that S_t admits an equivalent martingale measure in the filtration $\mathcal{F}^1$, and *a fortiori* in any smaller filtration. In other words, S_t is "inefficient" only insofar as it allows for time-varying expected returns.

2.1 Asset price dynamics

For informed agents the asset price dynamics (6) can be rewritten as:

$$dS_t = S_t((\mu_t - \lambda\sigma_t q U_t)dt + \sigma_t B_t^1)$$

where $B_t^1 = pW_t + qB_t$ is a Brownian Motion. Our first task is to establish a similar decomposition for S, or equivalently for the excess return Y, from the viewpoint of the uninformed agent, that is in terms of some $\mathcal{F}_t^0$-Brownian Motion B_t^0.

In principle, one could think of this problem as one of filtering, with the log fundamental value W_t playing the role of the signal, and the mispricing U_t the role of the noise. However, the analogy stops here, as the model does not fit the standard framework of filtering theory. Indeed, in usual filtering problems the signal is absolutely continuous, while our 'signal' W_t is a Brownian Motion, perturbed by the additive noise U_t, therefore common filtering techniques do not apply.

In a similarly naïve approach, one might hope to write Y as the solution of a stochastic differential equation of the form:

$$dY_t = \alpha(t, Y_t)dt + \beta(t, Y_t)dB_t^0$$

but this turns out to be impossible, since Y_t fails the Markov property in its natural filtration. To see this, recall that a Gaussian process has the Markov property if and only if its covariance function $\Gamma(s,t)$ satisfies the condition (see Proposition A.4):

$$\Gamma(s,t)\Gamma(t,u) = \Gamma(t,t)\Gamma(s,u) \qquad \text{for all } s \le t \le u \tag{8}$$

which fails for Y (except in the limit case $\lambda = 0$):

$$\mathrm{Cov}(Y_s, Y_t) = E\left[W_s W_t\right] + E\left[U_s U_t\right] = p^2(s \wedge t) + \frac{q^2}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda(t+s)}\right)$$

Furthermore, it is easily seen that the Markov property cannot be recovered even replacing the process U_t with a *stationary* Ornstein-Uhlenbeck process.

We have the following result, which is proved in section 5:

Theorem 2.1. *Let $Y_t = pW_t + qU_t$, and $(\mathcal{F}_t^0)_{t\in[0,\infty)}$ be the augmentation of the filtration generated by Y_t. Let $p, q, \lambda > 0$, $q^2 + p^2 = 1$. Denote by:*

$$\Gamma(s) = -\frac{1}{\lambda}\log(\cosh(\lambda p s) + p\sinh(\lambda p s)) \tag{9}$$

$$\gamma(s) = \Gamma'(s) = -p\tanh(\lambda p s + \operatorname{arc\,tanh} p) = \frac{1-p^2}{1+p\tanh(p\lambda s)} - 1 \tag{10}$$

Then the process

$$B_t^0 = Y_t + \int_0^t \left[\lambda\left(\gamma(s)+1\right)Y_s + \int_0^s \lambda^2\left(\gamma(s)+p^2\right)e^{\lambda(\Gamma(s)-\Gamma(u))}Y_u du\right]ds = \tag{11}$$

$$= Y_t + \lambda\int_0^t\int_0^s e^{\lambda(\Gamma(s)-\Gamma(u))}(1+\gamma(u))dY_u ds \tag{12}$$

is a Brownian Motion. The semimartingale decomposition of Y_t *under* $\mathcal{F}^0$ *is:*

$$Y_t = B_t^0 - \lambda \int_0^t e^{-\lambda(t-s)} \left(B_s^0 + \int_0^s \gamma(u) dB_u^0 \right) ds = \tag{13}$$

$$= B_t^0 - \lambda \int_0^t \int_0^s e^{-\lambda(s-u)} (1 + \gamma(u)) dB_u^0 ds \tag{14}$$

and its canonical representation is:

$$Y_t = \int_0^t \left(e^{-\lambda(t-s)} (1 + \gamma(s)) - \gamma(s) \right) dB_s^0 \tag{15}$$

We can read Theorem 2.1 as follows: for the uninformed agent, the asset price dynamics is:

$$dS_t = S_t((\mu_t + \sigma_t \nu_t) dt + \sigma_t dB_t^0) \tag{16}$$

where

$$\nu_t = -\lambda \left(\gamma(t) + 1 \right) Y_t - \lambda^2 \int_0^t \left(\gamma(s) + p^2 \right) e^{\lambda(\Gamma(t) - \Gamma(u))} Y_u du = \tag{17}$$

$$= -\lambda \int_0^t e^{-\lambda(t-u)} (1 + \gamma(u)) dB_u^0 \tag{18}$$

(16) and (17) show that for the uninformed agent, the drift $\mu_t + \sigma_t \nu_t$ depends on the entire past history of the price S_t. This suggests that optimal trading strategies will also involve past prices, a practice usually referred to as *technical analysis*. The next section precisely addresses this issue in the case of logarithmic utility.

3 Logarithmic Utility maximization

We now assume that both informed and uninformed agents invest in the market model described above in the time interval $[0, T]$, so as to maximize their expected utility from terminal wealth. Since we are going to consider logarithmic utility, similar conslusions will hold true for the problem of utility maximization from intertemporal consumption (cfr. the proof of Theorem 3.1 below). The discounted portfolio value at time t of an agent starting with initial capital x and holding H_t shares of the asset S, is given by:

$$X_t = x + (H \cdot S)_t \tag{19}$$

Agents are constrained to use *admissible* strategies, which must be predictable in their respective filtrations $(\mathcal{F}_t^i)_{i=0,1}$:

$$\mathcal{A}_x^i = \left\{ H \ \mathcal{F}^i\text{-predictable, } S\text{-integrable, } x + (H \cdot S)_t \geq 0 \text{ a.s. } \forall\, t \in [0, T] \right\} \ i = 0, 1$$

and solve the usual problem of utility maximization from terminal wealth:

$$\max \left\{ E\left[U(X_T)\right] : H \in \mathcal{A}_x^i \right\} \quad i = 0, 1 \tag{20}$$

The respective value functions are defined by:

$$u^i(x) = \sup \left\{ E\left[U(X_T)\right] : H \in \mathcal{A}_x^i \right\} \quad i = 0, 1 \tag{21}$$

Since we aim at explicit solutions, we shall consider the logarithmic utility function $U(x) = \log x$, as in [PK96, AIS98, Bau02, CIKHN04]. For this utility function, it is convenient to write the portfolio value in exponential form. We denote by $\pi = \frac{HS}{X}$ the fraction of wealth in the risky asset, and observe that (19) can be rewritten as:

$$X_t = x \exp\left(\int_0^t \left(\pi_s \mu_s - \pi_s^2 \frac{\sigma_s^2}{2} \right) ds + \int_0^t \pi_s \sigma_s dY_s \right) = \tag{22}$$

$$= x \exp\left(\int_0^t \pi_s \mu_s ds \right) \mathcal{E}\left((\pi\sigma) \cdot Y \right)_t \tag{23}$$

where $\mathcal{E}(X) = \exp\left(X - \frac{1}{2}\langle X \rangle\right)$ denotes the usual Doleans exponential.

We show the following:

Theorem 3.1. *For the logarithmic utility function:*

i) The value functions $u^i_{i=0,1}$ and the optimal strategies $\pi^i_{i=0,1}$ are given by:

$$\pi_t^0 = \frac{\mu_t + \sigma_t \nu_t}{\sigma_t^2} \qquad u^0(x) = \log x + \frac{1}{2} E\left[\int_0^T \frac{(\mu_t + \sigma_t \nu_t)^2}{\sigma_t^2} dt \right] \tag{24}$$

$$\pi_t^1 = \frac{\mu_t - \lambda \sigma_t q U_t}{\sigma_t^2} \qquad u^1(x) = \log x + \frac{1}{2} E\left[\int_0^T \frac{(\mu_t - \lambda \sigma_t q U_t)^2}{\sigma_t^2} dt \right] \tag{25}$$

where ν_t is given by (17).

ii) We have the following asymptotics, as $T \to \infty$:

$$u^0(x) \sim \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} dt + \frac{\lambda}{4}(1-p)^2 T \quad u^1(x) \sim \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} dt + \frac{\lambda}{4}(1-p^2) T \tag{26}$$

and therefore the additional logarithmic utility of the informed agent is given by:

$$u^1(x) - u^0(x) \sim \frac{\lambda}{2} p(1-p) T \tag{27}$$

Theorem 3.1 provides a quantitative assessment of the long-term impact of fads, for informed and uninformed agents, and immediately allows to perform some simple comparative statics. The additional logarithmic utility is proportional to the mean reversion speed λ: this is intuitively clear, because when temporary shocks are quickly absorbed, the informed agent can immediately profit from them, while the uninformed still ponders whether they are temporary of permanent.

The additional logarithmic utility is null in both cases $p = 0$ and $p = 1$. In the latter case the market is efficient, and there is no mean-reversion to be exploited. In the former case all shocks are temporary, therefore all agents can equally exploit mean-reversion, and there is no informational advantage. The highest utility gap is achieved for $(p, q) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, in which case the temporary shocks account for three quarters of the variance, and the permanent shocks for the remaining fourth.

To gain a better intuition of the quantities involved, we can also reformulate (24, 25) in terms of expected returns instead of additional utilities. More precisely, we ask the question: how much should the expected return increase, to compensate for the absence of mean-reversion? We denote (with a slight abuse of notation) by μ_0 and μ_1 the expected returns which make an uninformed (resp. informed) agent indifferent between a market with parameters $(\mu, \sigma, \lambda, p)$ and one with parameters $(\mu_0, \sigma, 0, p)$ (resp. $(\mu_1, \sigma, 0, p)$). From (24, 25) we have:

$$\mu_0 = \sqrt{\mu^2 + \frac{\lambda}{2}(1-p)^2\sigma^2} \qquad \mu_1 = \sqrt{\mu^2 + \frac{\lambda}{2}(1-p^2)\sigma^2} \tag{28}$$

These formulas highlight two other features: first, the equivalent additional returns $\mu^0 - \mu$ and $\mu^1 - \mu$ are decreasing in μ. When μ is low, mean-reversion generates a large fraction of expected utility, so its elimination requires a big increase in μ. Also, equivalent additional returns are increasing in σ: when volatility is high, the fraction of expected utility generated by the term $\frac{1}{2}\frac{\mu^2}{\sigma^2}T$ is low, and mean-reversion is again responsible for most of the expected utility.

Table 1 illustrates the above formulas for some typical parameter values (see section 4 as well as Poterba and Summers [PS88] and Shiller and Perron [SP85] for a discussion on 'typical' parameter values). In particular, it shows that for a wide range of values of μ and p, the equivalent additional return gap between uninformed and informed agents remains between 0.5% and 2%. Thus, in the metaphore of uninformed agents as *technical analysts* and of informed agents as *fundamental analysts*, we are tempted to conclude that the value of fundamental analysis is within the mentioned range. Needless to say, such an interpretation must be taken with care, as it hinges on many disputable assumptions, such as the fads model for asset prices, the choice of logarithmic utility, the long-term asymptotics approximation, and the plausibility of parameter values.

After discussing the economic interpretation of Theorem 3.1, let us look at its proof. To sketch the main idea, we first give a heuristic argument. Recall that the expected terminal logarithmic utility for a dynamics of the type:

$$dS_t = S_t(\alpha_t dt + \sigma_t dB_t)$$

	$p = 25\%$		$p = 50\%$		$p = 75\%$	
μ	$\mu^0 - \mu$	$\mu^1 - \mu$	$\mu^0 - \mu$	$\mu^1 - \mu$	$\mu^0 - \mu$	$\mu^1 - \mu$
0	3.98	5.13	2.65	4.59	1.33	3.51
1	3.10	4.23	1.83	3.70	0.66	2.65
2	2.45	3.51	1.32	3.01	0.40	2.04
3	1.98	2.95	1.00	2.49	0.28	1.62
4	1.64	2.51	0.80	2.09	0.21	1.32
5	1.39	2.17	0.66	1.79	0.17	1.11
6	1.20	1.90	0.56	1.56	0.14	0.95
7	1.05	1.68	0.49	1.37	0.12	0.83
8	0.93	1.51	0.43	1.22	0.11	0.74
9	0.84	1.36	0.38	1.10	0.10	0.66
10	0.76	1.24	0.35	1.00	0.09	0.60

Table 1: Equivalent additional returns (%) for uninformed ($\mu^0 - \mu$) and informed ($\mu^1 - \mu$) agents, for different values of the expected return μ and of the weight p of the fads component. The mean-reversion parameter λ is equal to 25%, which corresponds to a half time of fads of about three years, and volatility σ is set at 15%.

is generally given by the formula (cf. [PK96], [AIS98]):

$$\frac{1}{2}E\left[\int_0^T \frac{\alpha_t^2}{\sigma_t^2}dt\right] \tag{29}$$

For the informed agent the asset dynamics is:

$$\frac{dS_t}{S_t} =(\mu_t - \lambda q U_t \sigma_t)dt + \sigma_t B_t^1 \tag{30}$$

while for the uninformed agent, for large values of t (16) and (17) imply that:

$$\frac{dS_t}{S_t} \approx(\mu_t - \lambda(1-p)Y_t \sigma_t)dt + \sigma_t B_t^0 \tag{31}$$

If we apply formula (29) to (30) and (31), we formally obtain:

$$u^1(x) - u^0(x) \approx \frac{1}{2}\left(E\left[(\lambda q U_t)^2\right] - E\left[(\lambda(1-p)Y_t)^2\right]\right)T \approx \frac{\lambda}{2}p(1-p)T \tag{32}$$

Of course, this is just an heuristic argument, and a rigorous proof requires to show that formula (29) indeed applies to this case, and that the terms neglected in (31) and (32) are of an order smaller than T. This is precisely accomplished by the following:

Proof. In both cases of informed and uninformed agents we use the convex duality method in the version of Proposition A.2, hence we look for a probability measure Q and a strategy π which satisfies assumptions $i)-iv)$ in the mentioned Proposition. If Q is a probability equivalent to P, we introduce $Z^i_t = E\left[\frac{dQ}{dP}\middle|\mathcal{F}^i_t\right]$, which is a strictly positive P-martingale under $\mathcal{F}^i_t$, and therefore can be written as a Doleans exponential.

Let us first consider the case of informed agents. By the predictable representation property of the filtration $\mathcal{F}^1_t$ we may write Z^1 as:

$$Z^1 = \mathcal{E}\left(\theta \cdot W + \eta \cdot B\right) = \mathcal{E}\left(\theta \cdot W\right)\mathcal{E}\left(\eta \cdot B\right) \tag{33}$$

where the second equality follows from the independence of W and B. By Itô's formula, the Q-local martingale condition for S implies that:

$$\mu_s - \lambda\sigma_s q U_s + \sigma_s p\theta_s + \sigma_s q\eta_s = 0 \quad \text{a.s. in } dPdt \tag{34}$$

Then condition $ii)$ of Proposition A.2 becomes:

$$\frac{1}{x}\exp\left(-\int_0^T \pi_s\mu_s ds + \langle\pi\sigma\cdot Y\rangle_T\right)\mathcal{E}\left(-(\pi\sigma\cdot Y)\right)_T \;=\; y\mathcal{E}\left(\theta\cdot W\right)_T\mathcal{E}\left(\eta\cdot B\right)_T \tag{35}$$

If we solve the above equation simultaneously for all T, we obtain an a.s. equality between semimartingales. To achieve equality between the martingale parts, we need the following conditions:

$$\theta_s = -\,\pi_s\sigma_s p \tag{36}$$

$$\eta_s = -\,\pi_s\sigma_s q \tag{37}$$

Solving (34,36,37) for θ_s, η_s, π_s, we obtain

$$\pi_s = \frac{\mu_s - \lambda\sigma_s q U_s}{\sigma_s^2} \qquad \theta_s = -p\frac{\mu_s - \lambda\sigma_s q U_s}{\sigma_s} \qquad \eta_s = -q\frac{\mu_s - \lambda\sigma_s q U_s}{\sigma_s} \tag{38}$$

and it is easily seen that this is a solution of (35) with $y = 1/x$. To ensure that this π is the optimal strategy, it remains to check that $\frac{dQ}{dP} = Z^1_T$ defined by (33) with θ, η as in (38) indeed defines a probability density, and that assumptions $i)$ and $iii)$ in Proposition A.2 hold.

Z^1_T is a probability density if and only if Z^1 is martingale (and not merely a local martingale). To achieve this, we observe that both θ and η are Gaussian processes, and conclude by Proposition A.3 (a modification of the Novikov criterion for Gaussian integrands). Assumptions $i)$ and $iii)$ follow directly from Theorem 2.2 in [KS99].

For uninformed agents, the market is complete, and the only possible choice for Q is given by $Z^0_t = \mathcal{E}\left(\zeta \cdot B^0\right)$, where $\zeta_t = -(\frac{\mu_t}{\sigma_t} + \nu_t)$. As in the previous case, the martingale property of Z follows by Lemma A.3, observing that ν is a Gaussian process, and the other assumptions $i) - iv)$ follow similarly.

We now calculate the value functions u^1, u^0. We have:

$$u^i(x) = E\left[\log\frac{1}{yZ^i_T}\right] = \log x - E\left[\log Z^i_T\right] \qquad i = 0, 1$$

For Z^1, the second term can be calculated as:

$$E\left[\log Z_T^1\right] = E\left[(\theta \cdot W)_T + (\eta \cdot B)_T - \frac{1}{2}\int_0^T (\theta_s^2 + \eta_s^2)ds\right]$$

Since θ and η are Gaussian processes, the two stochastic integrals are uniformly integrable martingales by the Burkholder-Davis-Gundy inequalities, and their expectation is zero. Substituting (38) then provides the second part of (25), and integrating we obtain:

$$u^1(x) = \log x + \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2}dt + \lambda^2 q^2 \frac{1}{2}\int_0^T E\left[U_t^2\right]dt = \tag{39}$$

$$= \log x + \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2}dt + \frac{\lambda q^2}{4}T - \frac{1}{8}q^2(1 - e^{-2\lambda T}) \tag{40}$$

where he have used the fact that U_t has zero mean. Likewise, for $u^0(x)$ we have:

$$u^0(x) = \log x + \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2}dt + \frac{1}{2}\int_0^T E\left[\nu_t^2\right]dt \tag{41}$$

and the last term can be computed using (18):

$$\int_0^T E\left[\nu_t^2\right]dt = \lambda^2 \int_0^T \left(\int_0^t e^{-\lambda(t-u)}(1+\gamma(u))dB_u^0\right)^2 dt =$$

$$= \lambda^2 \int_0^T \int_0^t e^{-2\lambda(t-u)}(1+\gamma(u))^2 du dt =$$

$$= \frac{(1+p)^2 T\lambda}{2} - \log(1-p+e^{2pT\lambda}(1+p)) + \frac{1}{4}\left((-1+p^2)(1-e^{-2T\lambda}) + \log(16)\right) \sim$$

$$\sim \frac{1}{2}T\lambda(1-p)^2$$

which completes the proof. □

4 Estimation

We now apply 2.1 to the problem of estimation from real data. We consider the following model for discounted asset prices:

$$S_t = S_0 \exp\left(\left(\rho - \frac{1}{2}\right)\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dY_s\right) \tag{42}$$

where Y_t is defined by (5). This model is obtained by (7) assuming that the market price of risk $\rho = \frac{\mu_s}{\sigma_s^2}$ is constant.

Our data set consist of 925 monthly observations of the value-weighted and equal-weighted NYSE-AMEX indices of the CRSP data base for the 77-year

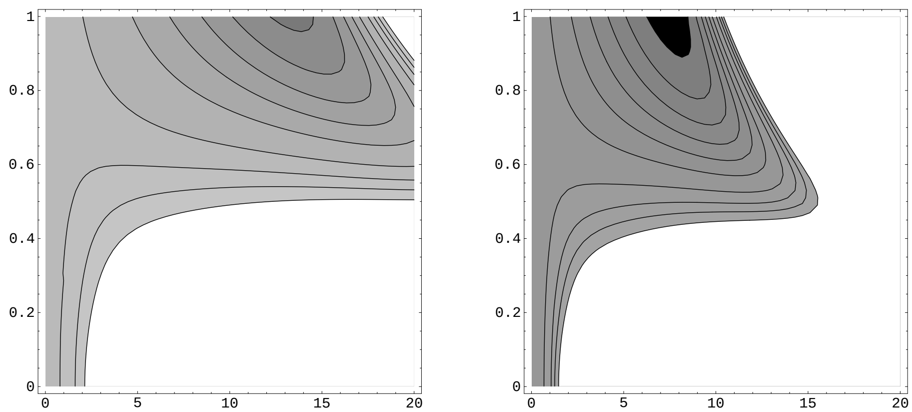

Figure 1: Significance levels of λ (horizontal axis) and p (vertical axis) for the value weighted (left) and equally weighted (right) NYSE-AMEX CRSP monthly indices over the period 1925:12-2002:12. The curves represent the significance levels (from white to black) 0.01, 0.025, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7.

period 1925:12-2002:12. To calculate discount factors we use CRSP monthly rates for Treasury bills over the same period.

Since we want to concentrate on λ and p, which are the parameters specific for the fads model under consideration, we start estimating ρ and σ_t with standard techniques. To control for weak forms of heteroscedasticity, we first apply a data transformation, which turns an equally spaced discrete-time observation from a heteroscedastic process into an unequally spaced discrete-time observation of an approximately homoscedastic process.

We subdivide the entire period into subintervals $[t_i, t_{i+1}]$ of 6 months each (using 12 months one obtains similar results). On each subinterval we calculate the usual maximum likelihood estimator $\hat{\sigma}_i$ for the volatility σ_t. Then, assuming σ_t constantly equal to $\hat{\sigma}_i$ on each subinterval $[t_i, t_{i+1}]$, we calculate the maximum likelihood estimator for market price of risk ρ, which is unique for the entire period. Then we define the sequence of times $\hat{t}_0 = 0$, $\hat{t}_{i+1} = \hat{t}_i + \hat{\sigma}_i^2(t_{i+1} - t_i)$, and the sequence of values $y_0 = 0$,

$$y_{\hat{t}_{i+1}} = y_{\hat{t}_i} + \frac{\log\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) - \hat{\rho}\ \hat{\sigma}_i^2(t_{i+1} - t_i)}{\hat{\sigma}_i}$$

In practice, we estimate the intrinsic time of the diffusion $\int_0^t \sigma_s dY_s$ from the data, and reconstruct Y_t at a smaller frequency. The approximation consists in estimating $\hat{t}_i$, and if one could calculate $\hat{t}_i$ exactly as $\int_0^{t_i} \sigma_s^2 ds$, the process $y_{\hat{t}_i}$ would be perfectly homoscedastic.

From the above procedure we obtain 155 observations, from which we estimate λ and p. To do this, we assume that $y_t = y_{\hat{t}_i}$ for $t \in [\hat{t}_i, \hat{t}_{i+1})$, and use first (53) then (45) to estimate a discrete-time observation of $\tilde{B}_t$. Note that one could have used (11) instead.

To check that this discretization is consistent, we have performed a Monte Carlo experiment, generating 10000 paths of the process (5). For each path, we estimated $\tilde{B}_t$ from (53) and (45), calculating the sample means of various functionals. Then we calculated the theoretical population means of the same functionals, and the sample means from the simulation of the same number of paths of a Brownian Motion. The three quantities obtained showed no statistically significant differences. All the *Mathematica* code used in this paper is available from the author upon request.

From the discrete-time observation $\tilde{B}_t$ (which depends on λ and p through (53) and (45)), the likelihood of the path is immediately calculated, and the results are shown in Figure 4. The white areas represent the pairs of parameter values which can can be rejected at a confidence level greater than 1%, and the curves represent level sets of the significance level.

The main message of the Figure 4 is the following: if the temporary component of price shocks q is to dominate the permanent component p, as the work of Le Roy and Porter [LP81] and Shiller [Shi81] on variance bounds suggest, then the speed of mean reversion must be very slow. In particular, λ must be roughly less than 2 at a 1% confidence level, which means that the half life of temporary shocks must be greater than 4 months. On the other hand, if mean reversion is slow then there is no way to discriminate from the data if the asset price follows a martingale ($p = 1$), an Ornstein-Uhlenbeck process ($p = 0$), or anything in between. This is shown by the fact that, in both pictures, the segment $\lambda = 0$ is contained in the confidence sets with levels $1\%, 2.5\%$ and 5% (as well as 10% for the equally weighted index).

5 Proof of Theorem 2.1

To prove Theorem 2.1 above, we need to introduce some preliminaries on L^2 kernels. We denote by $I = [0, +\infty)$, and by $L^2(I)$ and $L^2(I^2)$ the separable Hilbert spaces of real-valued, square-integrable functions, defined up to negligible sets of their respective Lebesgue measure.

There is a canonical isometry between the space of kernels $L^2(I^2)$ and the space of Hilbert-Schmidt operators from $L^2(I)$ to itself. More precisely, each kernel k induces the linear operator $K : L^2(I) \mapsto L^2(I)$ defined by

$$K : x \mapsto Kx \qquad (Kx)(t) = \int_0^t k(t,s)x(s)ds$$

and viceversa (see Dunford and Schwartz [DS88] for details). Given two such operators H and K, associated to the kernels h and k, we denote by HK their composition, which is easily seen to be associated to the kernel

$$\int_0^\infty h(t,u)k(u,s)du$$

The negative resolvent of an operator K is defined as the unique (if it exists) operator $\tilde{K}$ which satisfies the equality:

$$K + \tilde{K} = \tilde{K}K$$

Finally, we say that k is a Volterra kernel if $k(t,s) = 0$ for all $s > t$, and recall that if the operator K is associated to a Volterra kernel k, then its negative resolvent $\tilde{K}$ exists, and can be calculated via the Neumann series (see Theorem 2.7.1 in Smithies [Smi58]):

$$\tilde{K} = -\sum_{n=1}^{\infty} K^n \tag{43}$$

To study the dynamics of the process Y_t, it is useful to introduce the following auxiliary process, which has a simpler covariance structure:

Lemma 5.1. *Let Z_t be defined by:*

$$Z_t = qB_t + pW_t + \lambda p \int_0^t W_s ds \tag{44}$$

where $p, q, \lambda > 0$, and $p^2 + q^2 = 1$. We have that the process:

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s,u) dZ_u \right) ds \tag{45}$$

is a Brownian Motion, and the semimartingale decomposition of Z_t in its natural filtration is given by:

$$Z_t = B_t^0 - \lambda \int_0^t \int_0^s \gamma(u) dB_u^0 ds \tag{46}$$

where γ is defined by (10)*, and $g(t,s)$ is defined by:*

$$g(t,s) = \begin{cases} -\lambda\gamma(s)e^{\lambda(\Gamma(t)-\Gamma(s))} & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

Proof. It is clear that Z is a Gaussian process with zero mean. To compute its covariance, we can rewrite Z_t as:

$$Z_t = qB_t + p\int_0^t (1 + \lambda(t-u)) dW_u$$

Then it follows that:

$$\begin{aligned} \operatorname{Cov}(Z_t, Z_s) =& E\left[Z_t Z_s\right] = q^2 t \wedge s + p^2 \int_0^{t\wedge s} (1+\lambda(t-u))(1+\lambda(s-u)) du = \\ =& t \wedge s + p^2 \left(\lambda st + \lambda^2 (t\wedge s)^2 \left(\frac{1}{2}(t \vee s) - \frac{1}{6}(t \wedge s) \right) \right) = \\ =& t \wedge s + p^2 \int_0^t \int_0^s (\lambda + \lambda^2 u \wedge v) du dv = \\ =& t \wedge s - \int_0^t \int_0^s \tilde{f}(u,v) du dv \end{aligned}$$

where $\tilde{f}(u,v) = -p^2(\lambda + \lambda^2 u \wedge v)$.

We proceed by proving (46) first. From Proposition 2 in Hitsuda [Hit68] (or equation (16) in Cheridito [Che03]), it follows that if $\tilde{g}$ satisfies the following integral equation:

$$\tilde{f}(t,s) = \tilde{g}(t,s) - \int_0^s \tilde{g}(t,u)\tilde{g}(s,u)du \qquad \text{for } 0 \le s \le t \tag{48}$$

then there exists a Brownian Motion B_t^0 such that

$$Z_t = B_t^0 - \int_0^t \int_0^s \tilde{g}(s,u)dB_u^0 ds \tag{49}$$

is satisfied. We note that in our case:

$$\tilde{f}(t,s) = -p^2(\lambda + \lambda^2 t \wedge s) = -p^2(\lambda + \lambda^2 s) \qquad \text{for } 0 \le s \le t$$

therefore $\tilde{f}$ is independent of the first variable (for $0 \le s \le t$). This suggests to look for a solution of the form:

$$\tilde{g}(t,s) = \begin{cases} \lambda\gamma(s) & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

Then (48) becomes:

$$-p^2(\lambda + \lambda^2 s) = \lambda\gamma(s) - \lambda^2 \int_0^s \gamma(u)^2 du \qquad \text{for } 0 \le s \le t \tag{50}$$

In fact, (50) is clearly equivalent to the Cauchy problem:

$$\begin{cases} \gamma'(s) = \lambda(\gamma(s)^2 - p^2) \\ \gamma(0) = -p^2 \end{cases} \tag{51}$$

which admits the unique solution (10) (the inverse tangent is well-defined, as $p \in (0,1)$).

To prove (45), we use again the Hitsuda representation (see Theorems 1' and 2' in Hitsuda [Hit68] or Theorem 2 in Cheridito [Che03]), which states that B_t^0 can be written as:

$$B_t^0 = Z_t - \int_0^t \left(\int_0^s g(s,u)dZ_u \right) ds$$

where $g(t,s)$ denotes the negative resolvent of $\tilde{g}(t,s)$. Then an application of Lemma 5.2 yields (45). $\square$

Lemma 5.2. *Let $k \in L^2(I^2)$ the Volterra kernel defined by:*

$$k(t,s) = \begin{cases} \beta(t)\alpha(s) & \text{for } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases}$$

and K the associated Hilbert-Schmidt operator. Then its negative resolvent $\tilde{K}$ exists, and its kernel $\tilde{k}(t,s)$ is given by:

$$\tilde{k}(t,s) = \begin{cases} -k(t,s)e^{\int_s^t k(u,u)du} & \textit{for } 0 \leq s \leq t \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Since k is a Volterra kernel, then its negative resolvent exists and is given by (43). Denoting by k^n the kernel associated to K^n, we see by induction that:

$$k^n(t,s) = \begin{cases} \frac{\beta(t)\alpha(s)}{(n-1)!}\left(\int_s^t \beta(u)\alpha(u)du\right)^{n-1} & \text{for } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

For $n = 1$, (52) is trivial. Assuming it holds for n, for $0 \leq s \leq t$ we have that:

$$k^{n+1}(t,s) = \int_0^\infty k(t,u)k^n(u,s)du = \int_s^t \beta(t)\alpha(u)k^n(u,s)du =$$
$$= \frac{\beta(t)\alpha(s)}{(n-1)!}\int_s^t \beta(u)\alpha(u)\left(\int_s^u \beta(v)\alpha(v)dv\right)^{n-1} du = \frac{\beta(t)\alpha(s)}{n!}\left(\int_s^t \beta(u)\alpha(u)du\right)^n$$

which proves the inductive step. From (52) it follows that:

$$\tilde{k}(t,s) = -\sum_{n=0}^\infty \frac{\beta(t)\alpha(s)}{n!}\left(\int_s^t \beta(u)\alpha(u)du\right)^n = -k(t,s)e^{\int_s^t k(u,u)du} \quad \text{for } 0 \leq s \leq t$$

which concludes the proof. ∎

We can now proceed with the proof of Theorem 2.1:

Proof of Theorem 2.1. It is easily seen that:

$$Y_t = -\lambda\int_0^t Y_s ds + Z_t \tag{53}$$

where Z_t is defined by (44). We use Lemma 5.1, and substituting (53) in (45), we obtain that:

$$Y_t + \lambda\int_0^t Y_s ds = \int_0^t \int_0^s g(s,u)dY_u\, ds + \lambda\int_0^t \int_0^s g(s,u)Y_u du\, ds + B_t^0 \tag{54}$$

Integrating by parts, and denoting by $G(s,u) = e^{\lambda(\Gamma(s)-\Gamma(u))}$, we have that:

$$\int_0^s g(s,u)Y_u du = G(s,s)Y_s - \int_0^t G(s,u)dY_u \tag{55}$$

and substituting (55) in (54) we obtain (12). Then (11) can be obtained integrating (12) by parts. (14) follows applying Lemma 5.2 to (12). To prove (13), notice that (53) implies that:

$$Y_t = \int_0^t e^{-\lambda(t-s)} dZ_s \tag{56}$$

Substituting (46) in (56) we obtain:

$$Y_t = \int_0^t e^{-\lambda(t-s)} dB^0_s - \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \lambda\gamma(u) dB^0_u \right) ds \tag{57}$$

and integrating by parts the stochastic integral:

$$Y_t = B^0_t - \int_0^t \lambda e^{-\lambda(t-s)} \left(B^0_s + \int_0^s \gamma(u) dB^0_u \right) ds$$

which proves (13). Finally, to show (15), rewrite (57) as:

$$\begin{aligned} Y_t &= \int_0^t e^{-\lambda(t-s)} dB^0_s - \int_0^t \lambda\gamma(u) \left(\int_u^t e^{-\lambda(t-s)} ds \right) dB^0_u = \\ &= \int_0^t \left(e^{-\lambda(t-s)}(1+\gamma(s)) - \gamma(s) \right) dB^0_s \end{aligned}$$

□

A Appendix

The following is a classical duality criterion for optimality, in the spirit of [KLSX91] and [KS99]. Although the result is well-known, the assumptions used by different authors often differ slightly. We report the exact version used in this paper, along with its standard proof.

Assumption A.1. *A utility function $U : (0, \infty) \mapsto \mathbb{R}$ satisfies the* Inada conditions *if it is strictly increasing, strictly concave, continuously differentiable, and*

$$U'(0^+) = +\infty, \qquad U'(+\infty) = 0 \tag{58}$$

Proposition A.2. *Let $(S_t)_{t\in[0,T]}$ be a semimartingale, U a utility function satisfying Assumption A.1, and denote by $I = (U')^{-1}$.*

If there exists a probability Q equivalent to P, an admissible strategy $\hat{H}$ with terminal wealth $\hat{X}_T = x + (\hat{H} \cdot S)_T$, and a Lagrange multiplier $y > 0$ such that:

i) $E_Q\left[\hat{X}_T\right] = x$

ii) $\hat{X}_T = I(y\frac{dQ}{dP})$

iii) $E\left[V(y\frac{dQ}{dP})\right] < \infty$

iv) S *is a* Q*-local martingale on* $[0, T]$

Then $\hat{H}$ *is an optimizer for problem* (20)*, and the maximal expected utility is given by* $u(x) = E\left[U\left(I\left(y\frac{dQ}{dP}\right)\right)\right]$.

Proof. As usual, we denote the convex dual of U as:

$$V(y) = \sup_{x>0}\left(U(x) - xy\right) \qquad \text{for } y > 0$$

It is clear that:

$$U(X) \leq XY + V(Y) \qquad \text{for all } X, Y \geq 0$$

substituting $Y = y\frac{dQ}{dP}$ and $X = X_T = (H \cdot S)_T$ we have:

$$E\left[U(X_T)\right] \leq yE\left[X_T\frac{dQ}{dP}\right] + E\left[V\left(y\frac{dQ}{dP}\right)\right] \tag{59}$$

For any admissible strategy H, the gain process $(H \cdot S)$ is a Q-local martingale bounded from below, hence a supermartingale. This implies that:

$$E_Q\left[X_T\right] = E_Q\left[x + (H \cdot S)_T\right] \leq x \tag{60}$$

(59) and (60) provide an upper bound for the maximal expected utility, which is finite by assumption *iii*). It is sufficient to show that this bound is attained. In fact, by *i*) and *ii*) we have:

$$E\left[U(\hat{X}_T)\right] = yE_Q\left[\hat{X}_T\right] + E\left[V\left(y\frac{dQ}{dP}\right)\right] = xy + E\left[V\left(y\frac{dQ}{dP}\right)\right] \tag{61}$$

which concludes the proof. □

The following lemma is a substitute of the Novikov criterion for Gaussian integrands (cf. VIII.1.40 in [RY99] or 3.5.14 in [KS91]):

Lemma A.3. *Let* B *be a* $\mathcal{F}_t^0$*-Brownian Motion, and* H *a Gaussian process, adapted to* $\mathcal{F}_t^0$*. Then* $\mathcal{E}\left(H \cdot B\right)$ *is a martingale.*

Proof. Since $\mathcal{E}\left(H \cdot B\right)$ is a local martingale, we only need to prove that $E\left[\mathcal{E}\left(H \cdot B\right)_t\right] = 1$ for all t. Let $t_0 = 0 < t_1 < \ldots < t_n = t$ a partition of $[0, t]$. Then we have that:

$$E\left[\mathcal{E}\left(H \cdot B\right)_t\right] = E\left[\prod_{i=1}^{n} E\left[\left.\frac{\mathcal{E}\left(H \cdot B\right)_{t_i}}{\mathcal{E}\left(H \cdot B\right)_{t_{i-1}}}\right| \mathcal{F}_{t_{i-1}}^0\right]\right] \tag{62}$$

and it suffices to show that for some partition $(t_i)_{i=0}^n$, each of the conditional expectations in the right-hand side is equal to 1. To this purpose, we choose an equally spaced partition, with $t_i - t_{i-1} = \delta$ for all i, and apply the Novikov criterion to the local martingales $M_s^i = \mathcal{E}(H \cdot B)_{t_{i-1}+s} / \mathcal{E}(H \cdot B)_{t_{i-1}}$ on $[0, t_i - t_{i-1}]$. We have that:

$$E\left[\exp\left(\frac{1}{2}\left\langle M^i\right\rangle_{t_i - t_{i-1}}\right)\right] = E\left[\exp\left(\frac{1}{2}\int_{t_{i-1}}^{t_i} H_s^2 ds\right)\right] \leq$$

$$\leq \frac{1}{\delta}\int_{t_{i-1}}^{t_i} E\left[\exp\left(\frac{\delta}{2} H_s^2\right)\right] ds \quad (63)$$

where the last inequality follows from Jensen's. Since H_s is a Gaussian random variable, $E\left[\exp\left(\frac{\delta}{2} H_s^2\right)\right] < \infty$ if and only if $\delta \operatorname{Var}(H_s) < 1$, therefore it suffices to choose $\delta < 1/(\sup_{s\in[0,t]} \operatorname{Var}(H_s))$. This completes the proof.

□

The following is a necessary and sufficient condition for a Gaussian process to be Markov (see III.1.13 in [RY99]):

Proposition A.4. *A centered Gaussian process $(X_t)_{t\geq 0}$ with covariance function Γ is Markov if and only if, for all $s \leq t \leq u$, we have:*

$$\Gamma(s,u)\Gamma(t,t) = \Gamma(s,t)\Gamma(t,u)$$

Proof. The claim is trivial when $\Gamma(t,t)$=0. Otherwise, since a Gaussian process is determined by its covariance structure, the Markov property is equivalent to:

$$E\left[X_s X_u \left| X_t\right.\right] = E\left[X_s \left| X_t\right.\right] E\left[X_u \left| X_t\right.\right]$$

for all $s \leq t \leq u$. The conditional expectations in the right-hand side can be calculated as projections:

$$E\left[X_s \left| X_t\right.\right] = \frac{E\left[X_s X_t\right]}{E\left[X_t^2\right]} X_t = \frac{\Gamma(s,t)}{\Gamma(t,t)} X_t \qquad E\left[X_u \left| X_t\right.\right] = \frac{\Gamma(u,t)}{\Gamma(t,t)} X_t$$

It follows that:

$$\Gamma(s,u) = E\left[X_s X_u\right] = E\left[E\left[X_s X_u \left| X_t\right.\right]\right] = E\left[\frac{\Gamma(s,t)\Gamma(t,u)}{\Gamma(t,t)^2} X_t^2\right] = \frac{\Gamma(s,t)\Gamma(t,u)}{\Gamma(t,t)}$$

which concludes the proof. □

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